Statistical Inference of Modules in Large-Scale Networks

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Universität Bremen

Oldenburg, November 2013

NETWORKS

Networks form the substrate of a wide variety of complex systems.

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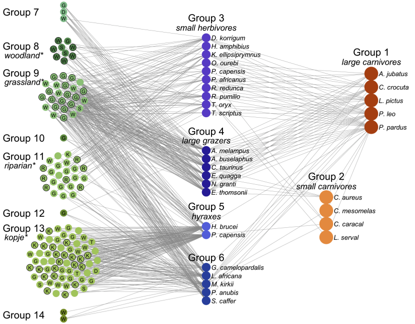
- Food webs



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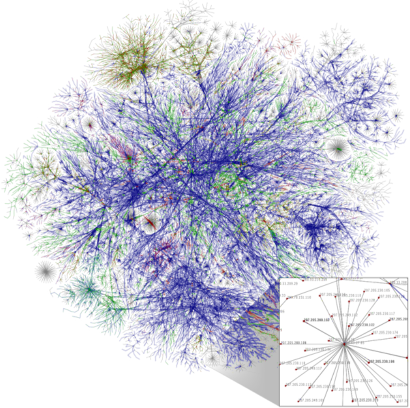

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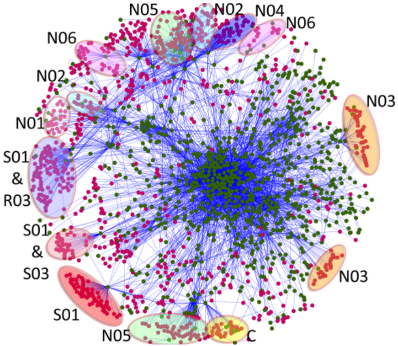

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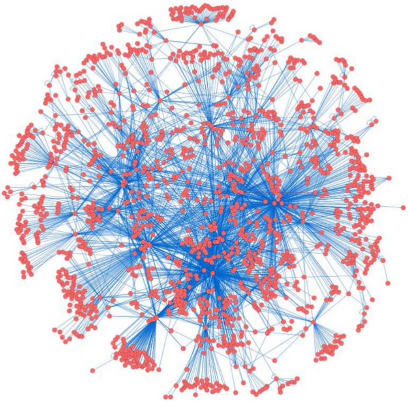

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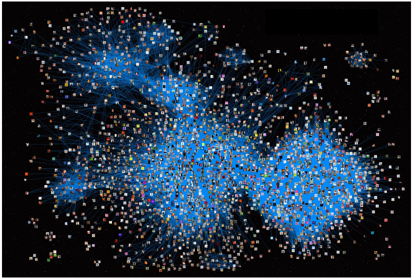

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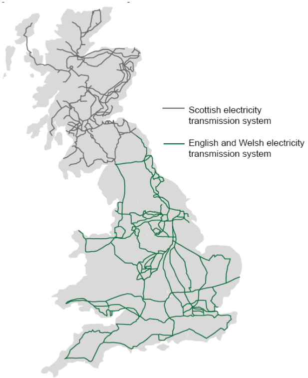

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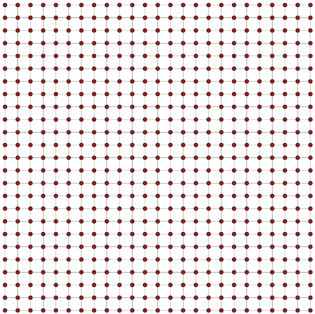
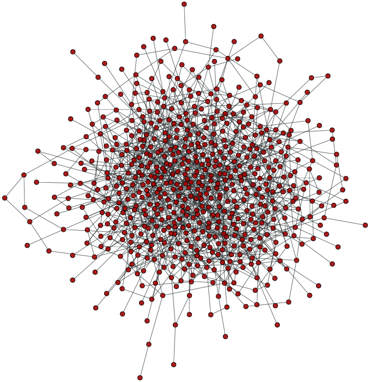
Structure $\leftrightarrow$ Dynamics $\leftrightarrow$ Evolution

Structure of Networks

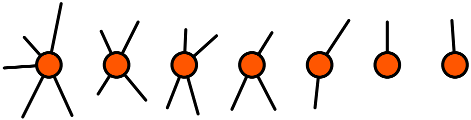
Somewhere between regular and random



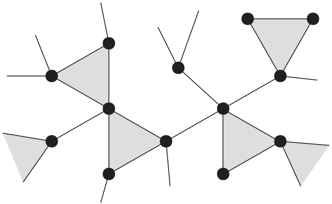


Local (or small-scale) structure

Degree distribution

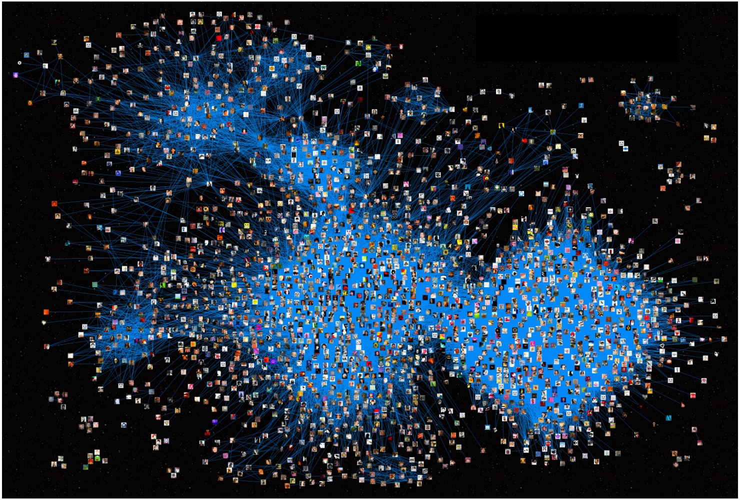


Clustering (Triangles)

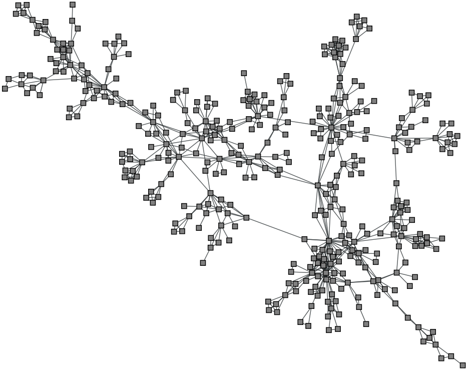


Global (or large-scale) structure

Modular structure



Levels of description



Real Network
(Network scientists)

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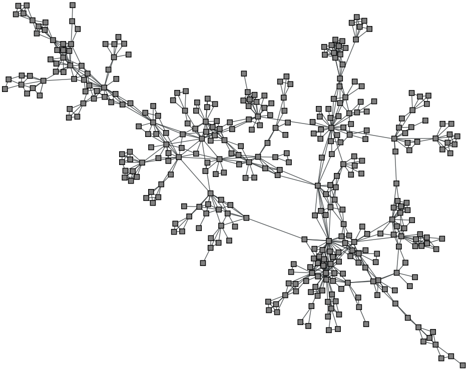

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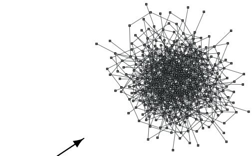

Random graph
(no structure)

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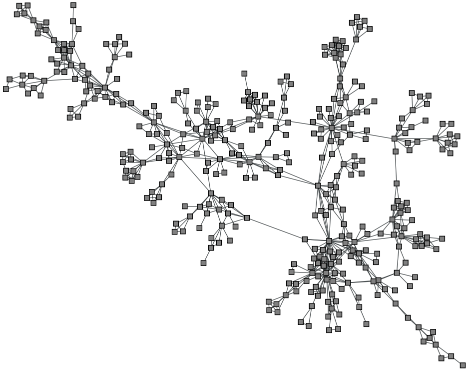

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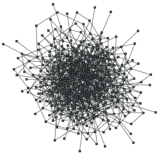

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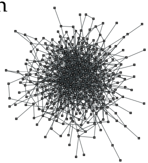

Degree
distribution

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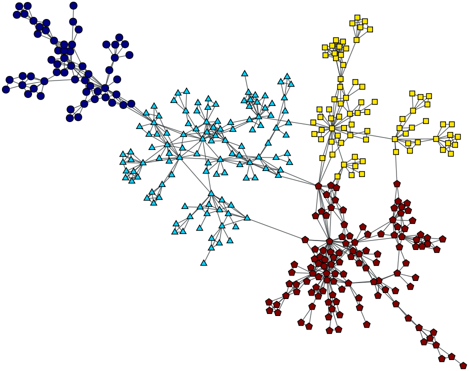

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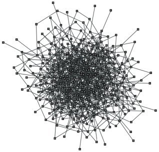

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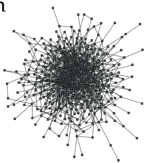

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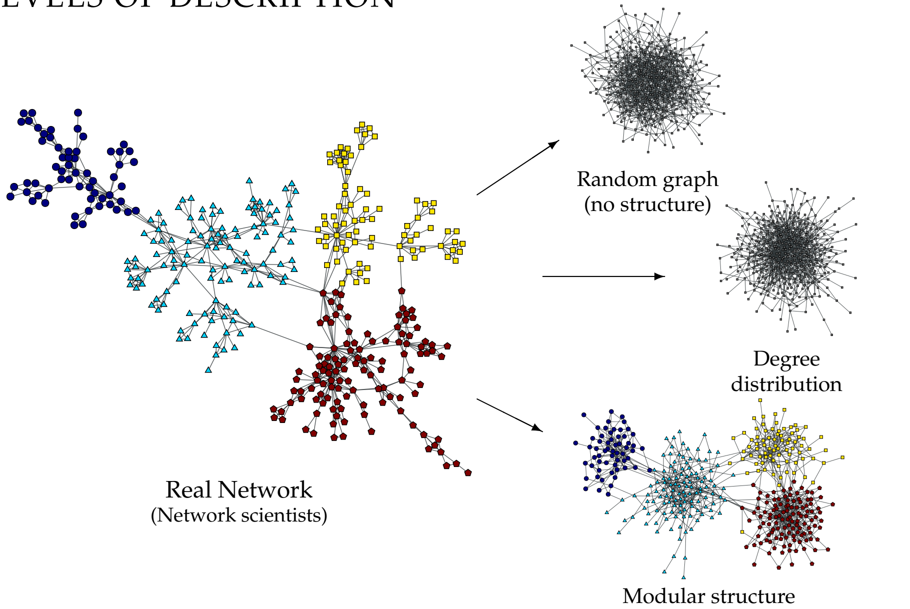
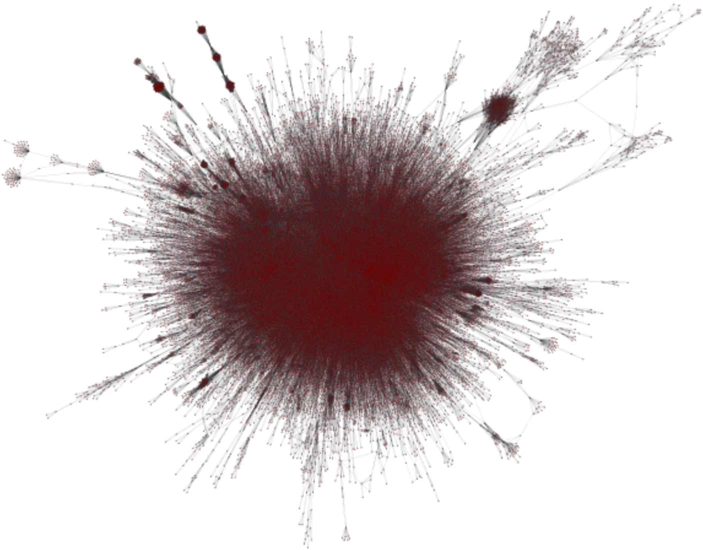
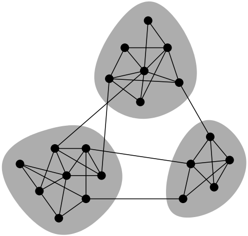


Problem: How to detect and characterize modular structure?

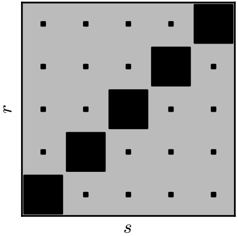


Simpler approach: Community Structure

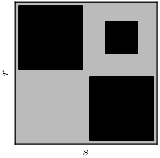
Focus on one of many possible patterns: *Communities*



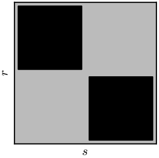
Communities



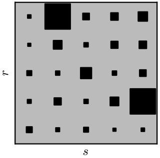
Excludes: Core-periphery, bipartite, multipartite, ...



Core-periphery



Bipartite



Arbitrary

Community Structure: Modularity maximization.

Idea: Find the partition of nodes, such that the fraction of internal edges is higher than expected given a *null model*.

$$Q = \frac{1}{2E} \sum_{ij} (A_{ij} - p_{ij}) \delta_{b_i, b_j}$$

$$A_{ij} = \begin{cases} 1, & \text{if } i, j \text{ are connected} \\ 0, & \text{otherwise.} \end{cases}$$

$$p_{ij} = \frac{k_i k_j}{2E} \rightarrow \text{Random graph}$$

Task: Find the partition $\{b_i\}$ which maximizes Q.

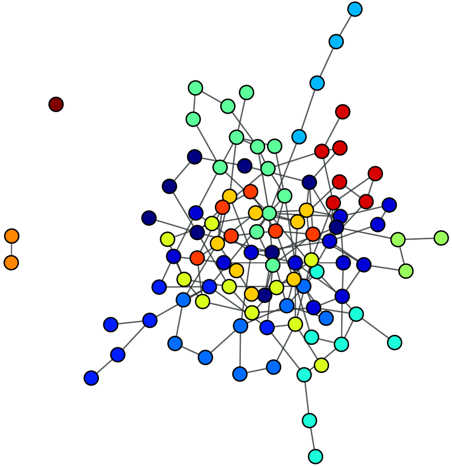
Newman and Girvan, Phys. Rev. E 69 026113 (2004).

Problems with modularity

- Restricted to community structure.
- Maximization of Q is difficult (NP-Complete).
- No built-in validation: Cannot distinguish structure from noise.
- Lack of consistency: Many different partitions with a similar Q.
- Resolution limit: Small modules cannot be detected.

Modularity: No built-in validation

A fully random graph:

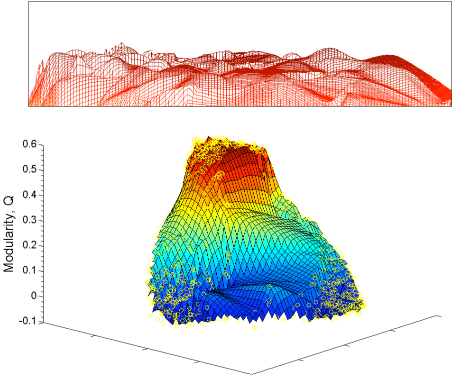


$$Q \simeq 0.55$$

Guimerà et al, Phys. Rev. E 70, 025101 (2004)

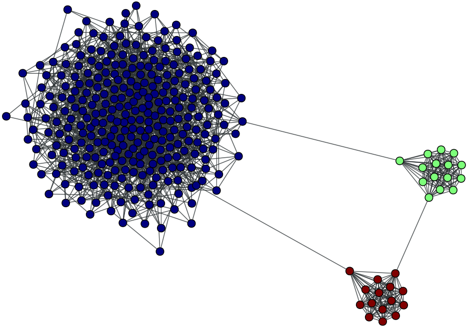
Modularity: Lack of consistency

Partitions are often degenerate.



Good et al, Phys. Rev. E 81, 046106 (2010)

Modularity: Resolution limit



The two smaller communities are merged together if $e_c < \sqrt{E/2}$

(Fortunato et al, PNAS 2007)

Characteristic block size: $\sim \sqrt{E}$

Distorted picture: Heavily skewed towards homogeneous blocks.

Why does modularity fail?

Global model? Wrong null model?

The main problem: It is a rather *ad hoc* approach.

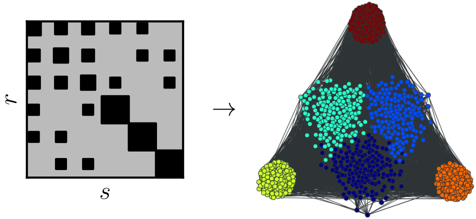
> Comparison with the null model is not connected with statistical significance.

A better approach: Stochastic blockmodels

Statistical inference of *generative models*

Traditional: N nodes divided into B blocks.

Parameters: $b_i \rightarrow$ block membership of node i
$e_{rs} \rightarrow$ number of edges from block r to s.



Degree-corrected: Arbitrary degree sequence: $\{k_i\}$

P. W. Holland et al., Soc Networks 5, 109 (1983)
B. Karrer and M. Newman, Phys. Rev. E 83, 016107 (2011)

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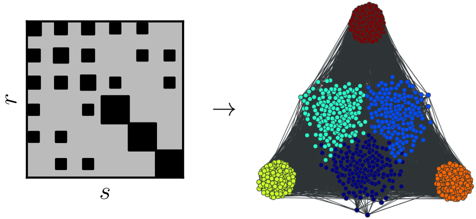

Degree-corrected: Arbitrary degree sequence: $\{k_i\}$

- Not restricted to assortative structures ("communities"). Could be bipartite, multipartite, core-periphery, etc.
- Easily modifiable for directed graphs.
- Inference $\to$ Maximize posterior probability $\mathcal{P}(G|\{e_{rs}\}, \{b_i\})$

P. W. Holland et al., Soc Networks 5, 109 (1983)
B. Karrer and M. Newman, Phys. Rev. E 83, 016107 (2011)

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T.P.P., PRE 85, 056122 (2012)

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Inference ↔ Compression

$$\max_{\{e_{rs}\},\{b_i\}} \ln\mathcal{P} \equiv \min_{\{e_{rs}\},\{b_i\}} \mathcal{S}$$

→ Minimization of information required to describe the network, *when the model is known*.

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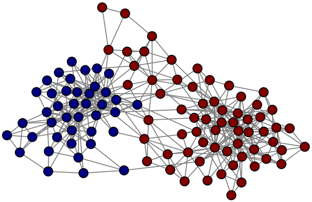

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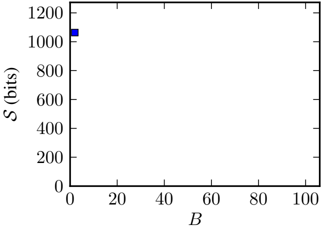

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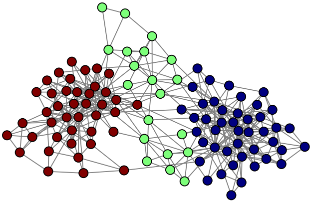

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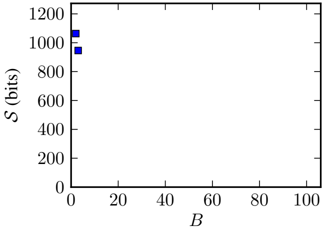

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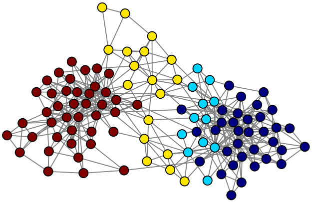
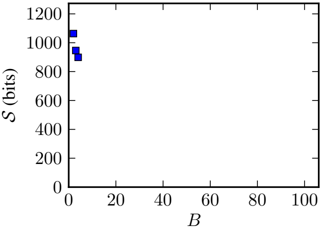


$B = 4$



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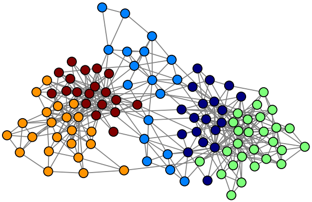

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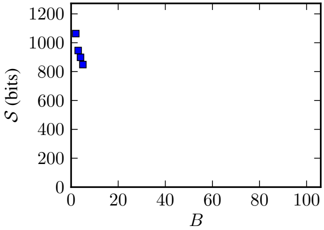

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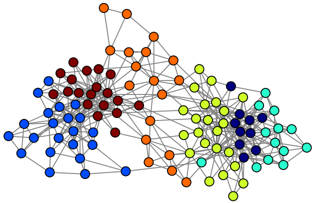

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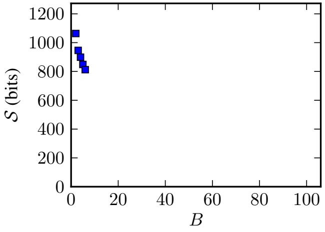

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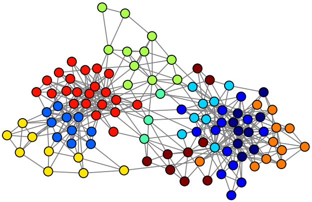
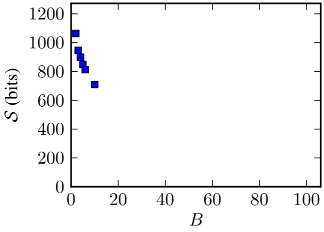


$B = 10$



Maximum likelihood

Microcanonical formulation: $\mathcal{P}(G|\{e_{rs}\},\{b_i\}) = \frac{1}{\Omega(\{e_{rs}\},\{b_i\})}$

Ensemble entropy: $\mathcal{S}(\{e_{rs}\},\{b_i\}) = \ln\Omega(\{e_{rs}\},\{b_i\}) = -\ln\mathcal{P}(G|\{e_{rs}\},\{b_i\})$

$$\mathcal{S} \cong -E - \sum_k N_k \ln k! - \frac{1}{2}\sum_{rs} e_{rs} \ln\left(\frac{e_{rs}}{e_r e_s}\right)$$

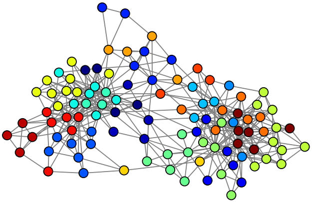
$$\max_{\{e_{rs}\},\{b_i\}} \ln\mathcal{P} \equiv \min_{\{e_{rs}\},\{b_i\}} \mathcal{S}$$

$\rightarrow$

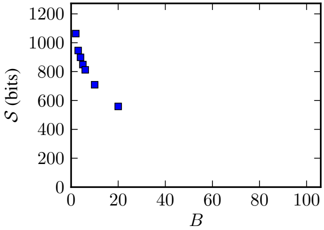
Inference $\leftrightarrow$ Compression

Minimization of information required to describe the network, *when the model is known*.

Works like a charm if the number of blocks B is known...



$B = 20$



Maximum likelihood

Microcanonical formulation: $\mathcal{P}(G|\{e_{rs}\},\{b_i\}) = \dfrac{1}{\Omega(\{e_{rs}\},\{b_i\})}$

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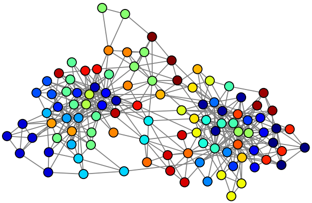
$$\max_{\{e_{rs}\},\{b_i\}} \ln\mathcal{P} \equiv \min_{\{e_{rs}\},\{b_i\}} \mathcal{S}$$

$\rightarrow$

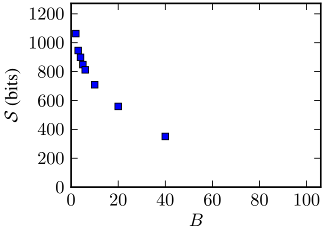
Inference $\leftrightarrow$ Compression

Minimization of information required to describe the network, *when the model is known*.

Works like a charm if the number of blocks B is known...



$B = 40$



T.P.P., PRE 85, 056122 (2012)

Maximum likelihood

Microcanonical formulation: $\mathcal{P}(G|\{e_{rs}\},\{b_i\}) = \dfrac{1}{\Omega(\{e_{rs}\},\{b_i\})}$

Ensemble entropy: $\mathcal{S}(\{e_{rs}\},\{b_i\}) = \ln\Omega(\{e_{rs}\},\{b_i\}) = -\ln\mathcal{P}(G|\{e_{rs}\},\{b_i\})$

$$\mathcal{S} \cong -E - \sum_k N_k \ln k! - \frac{1}{2}\sum_{rs} e_{rs}\ln\left(\frac{e_{rs}}{e_r e_s}\right)$$

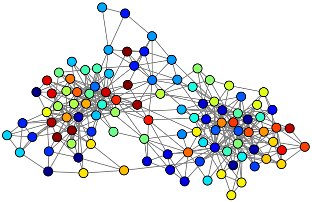
$$\max_{\{e_{rs}\},\{b_i\}} \ln\mathcal{P} \equiv \min_{\{e_{rs}\},\{b_i\}} \mathcal{S}$$

→

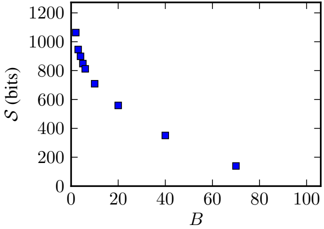
Inference ↔ Compression

Minimization of information required to describe the network, *when the model is known*.

Works like a charm if the number of blocks B is known...



$B = 70$



T.P.P., PRE 85, 056122 (2012)

Maximum likelihood

Microcanonical formulation: $\mathcal{P}(G|\{e_{rs}\},\{b_i\}) = \dfrac{1}{\Omega(\{e_{rs}\},\{b_i\})}$

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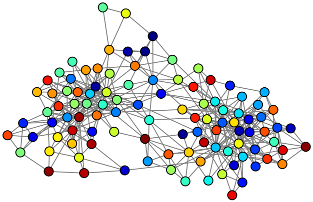
$$\max_{\{e_{rs}\},\{b_i\}} \ln\mathcal{P} \equiv \min_{\{e_{rs}\},\{b_i\}} \mathcal{S}$$

→

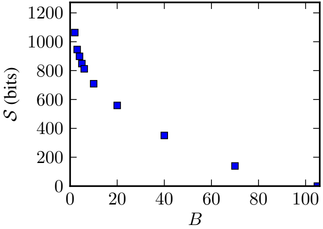
Inference ↔ Compression

Minimization of information required to describe the network, *when the model is known*.

Works like a charm if the number of blocks B is known...



$B = N$



... otherwise: **Overfitting!**

T.P.P., PRE 85, 056122 (2012)

Solution: Minimum Description Length

$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

Solution: Minimum Description Length

$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

SOLUTION: MINIMUM DESCRIPTION LENGTH

$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

$$\mathcal{L} = \ln \left(\!\! \binom{\left(\!\binom{B}{2}\!\right)}{E} \!\! \right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

Solution: Minimum Description Length

$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

$$\mathcal{L} = \ln \left(\!\! \binom{\left(\!\binom{B}{2}\!\right)}{E} \!\! \right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!

Solution: Minimum Description Length

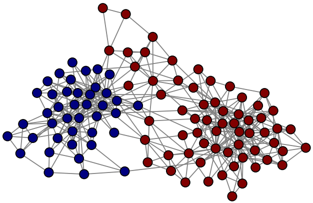
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

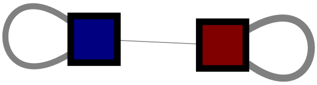
$$\mathcal{L} = \ln \left(\!\!\binom{\left(\!\binom{B}{2}\!\right)}{E}\!\!\right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 2, \mathcal{S} \simeq 1805.3$ bits



Model, $\mathcal{L} \simeq 122.6$ bits

$\Sigma \simeq 1926.9$ bits

Solution: Minimum Description Length

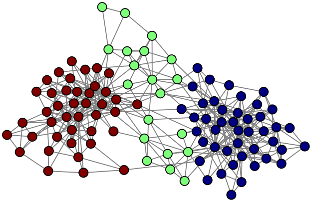
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

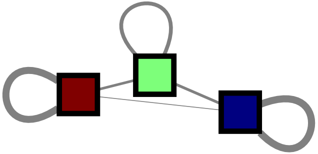
$$\mathcal{L} = \ln \left(\!\!\binom{\left(\!\binom{B}{2}\!\right)}{E}\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 3$, $\mathcal{S} \simeq 1688.1$ bits



Model, $\mathcal{L} \simeq 203.4$ bits

$\Sigma \simeq 1891.6$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

SOLUTION: MINIMUM DESCRIPTION LENGTH

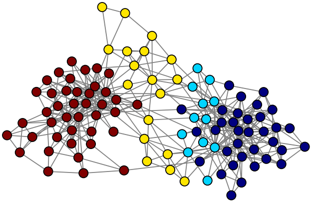
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

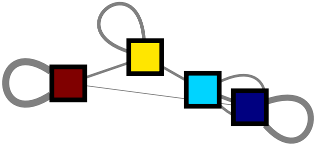
$$\mathcal{L} = \ln \left(\!\!\binom{\binom{B}{2}}{E}\!\!\right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 4$, $\mathcal{S} \simeq 1640.8$ bits



Model, $\mathcal{L} \simeq 270.7$ bits

$\Sigma \simeq 1911.5$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

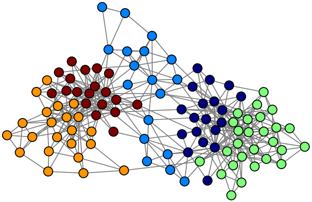
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

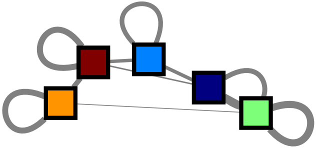
$$\mathcal{L} = \ln \left(\!\!\left(\begin{array}{c} \left(\!\binom{B}{2}\!\right) \\ E \end{array} \right)\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 5$, $\mathcal{S} \simeq 1590.5$ bits



Model, $\mathcal{L} \simeq 330.8$ bits

$\Sigma \simeq 1921.3$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

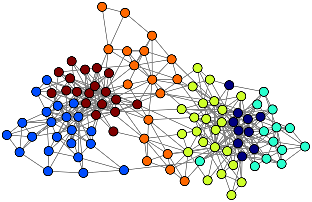
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

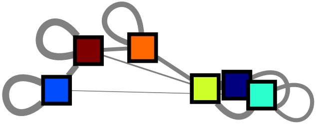
$$\mathcal{L} = \ln \left(\!\!\binom{\binom{B}{2}}{E}\!\!\right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 6$, $\mathcal{S} \simeq 1554.2$ bits



Model, $\mathcal{L} \simeq 386.7$ bits

$\Sigma \simeq 1940.9$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

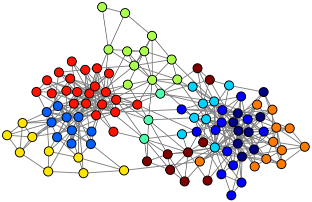
$\mathcal{S} \to$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \to$ Information required to describe *the model*.

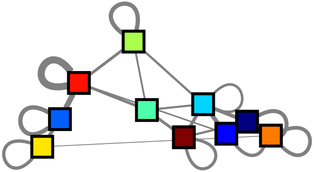
$$\mathcal{L} = \ln\left(\!\!\left(\begin{matrix}\left(\!\binom{B}{2}\!\right)\\ E\end{matrix}\right)\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N\sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 10$, $\mathcal{S} \simeq 1451.0$ bits



Model, $\mathcal{L} \simeq 590.8$ bits

$\Sigma \simeq 2041.8$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

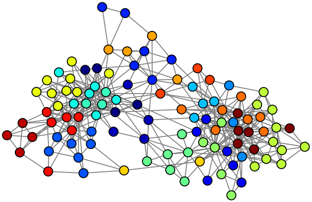
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

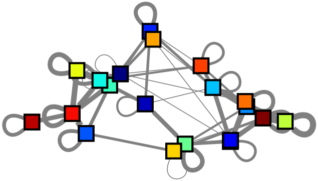
$$\mathcal{L} = \ln \left(\!\!\binom{\binom{B}{2}}{E}\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 20$, $\mathcal{S} \simeq 1300.7$ bits



Model, $\mathcal{L} \simeq 1037.8$ bits

$\Sigma \simeq 2338.6$ bits

Solution: Minimum Description Length

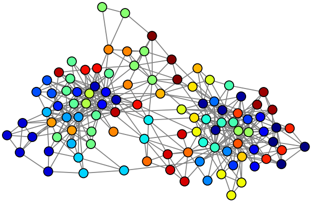
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

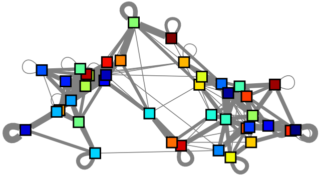
$$\mathcal{L} = \ln \left(\!\!\binom{\left(\!\binom{B}{2}\!\right)}{E}\!\!\right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 40$, $\mathcal{S} \simeq 1092.8$ bits



Model, $\mathcal{L} \simeq 1730.3$ bits

$\Sigma \simeq 2823.1$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

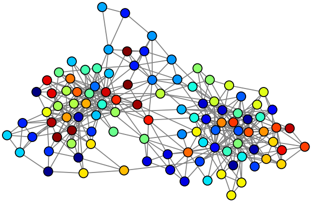
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

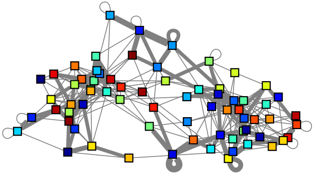
$$\mathcal{L} = \ln \left(\!\!\binom{\left(\!\binom{B}{2}\!\right)}{E}\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 70$, $\mathcal{S} \simeq 881.3$ bits



Model, $\mathcal{L} \simeq 2427.3$ bits

$\Sigma \simeq 3308.7$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

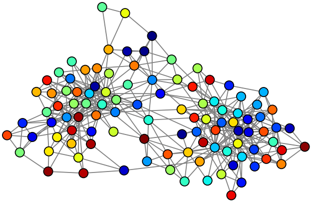
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

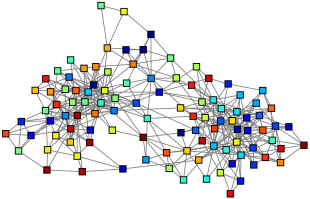
$$\mathcal{L} = \ln \left(\!\!\binom{\left(\!\binom{B}{2}\!\right)}{E}\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$\Sigma = \mathcal{S} + \mathcal{L}$

Total information necessary, without a priori knowledge of the model!



$B = N$, $\mathcal{S} = 0$ bits



Model, $\mathcal{L} \simeq 2973.0$ bits

$\Sigma \simeq 3714.9$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

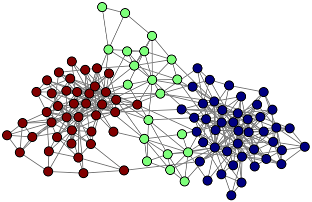
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

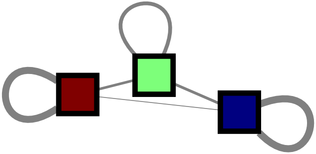
$$\mathcal{L} = \ln \left(\!\!\left({\left(\!\binom{B}{2}\!\right) \atop E} \right)\!\!\right) + \ln N! - \sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$$\Sigma = \mathcal{S} + \mathcal{L}$$

Total information necessary, without a priori knowledge of the model!



$B = 3$, $\mathcal{S} \simeq 1688.1$ bits



Model, $\mathcal{L} \simeq 203.4$ bits

$\Sigma \simeq 1891.6$ bits

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Solution: Minimum Description Length

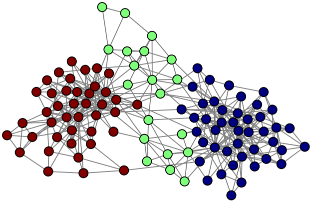
$\mathcal{S} \rightarrow$ Information required to describe the network, *when the model is known*.

$\mathcal{L} \rightarrow$ Information required to describe *the model*.

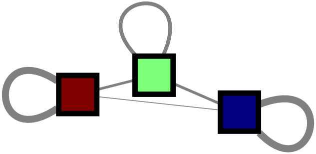
$$\mathcal{L} = \ln \left(\!\!\binom{\binom{B}{2}}{E}\!\!\right) + \ln N! - \textstyle\sum_r \ln n_r! - N \sum_k p_k \ln p_k$$

$\Sigma = \mathcal{S} + \mathcal{L}$

Total information necessary, without a priori knowledge of the model!

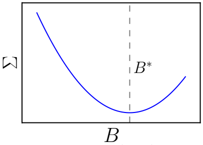


$B = 3$, $\mathcal{S} \simeq 1688.1$ bits



Model, $\mathcal{L} \simeq 203.4$ bits

$\Sigma \simeq 1891.6$ bits



$\rightarrow$

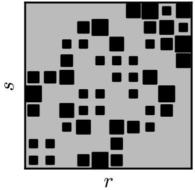
Occam's razor

The best model is the one which most compresses the data.

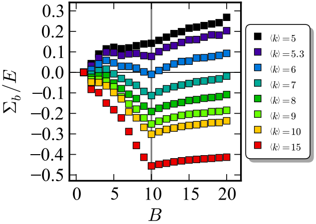
T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

WORKS VERY WELL... IF THE BLOCK STRUCTURE IS DETECTABLE!

Generated ($B = 10$)



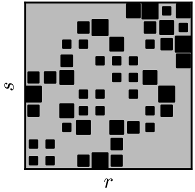
Planted



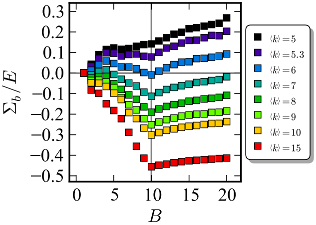
T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

WORKS VERY WELL... IF THE BLOCK STRUCTURE IS DETECTABLE!

Generated ($B = 10$)

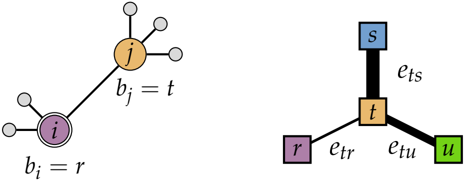


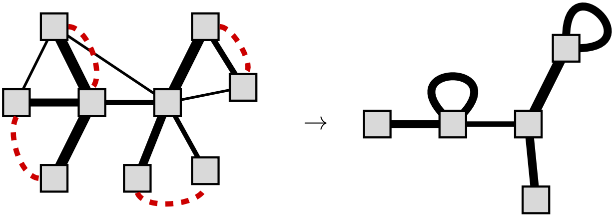
Planted



Advantage of MDL: Very efficient

Scalable algorithm: (fast) MCMC / Greedy Aggomeration





arXiv:1310.4378

Total running time: $O(N \ln^2 N)$

EXAMPLE: THE INTERNET MOVIE DATABASE (IMDB)

Bipartite network of actors and films.
Fairly large: $N = 372,787$, $E = 1,812,657$

Example: The Internet Movie Database (IMDB)

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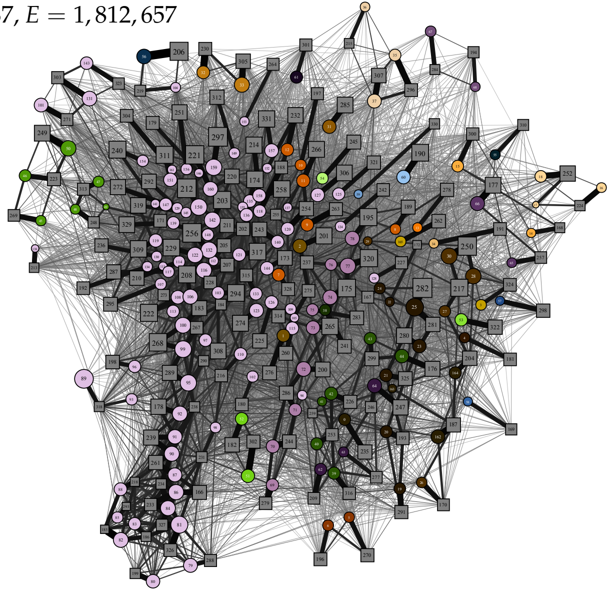

T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

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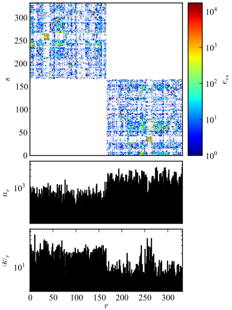
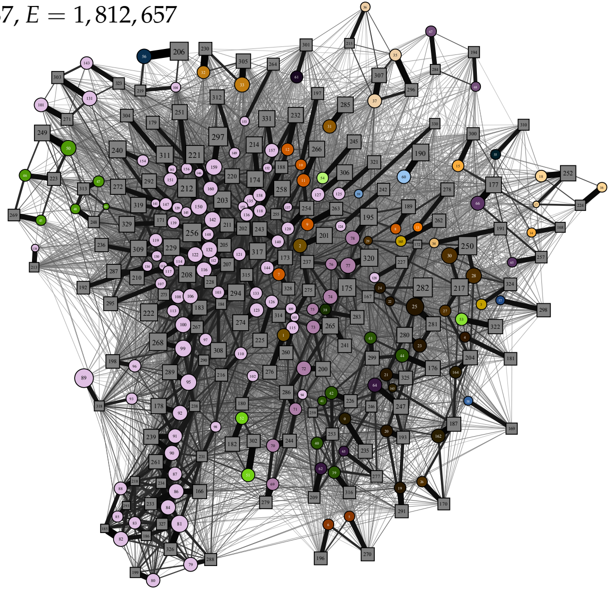


Bipartiteness is fully uncovered!



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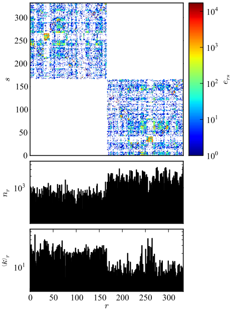

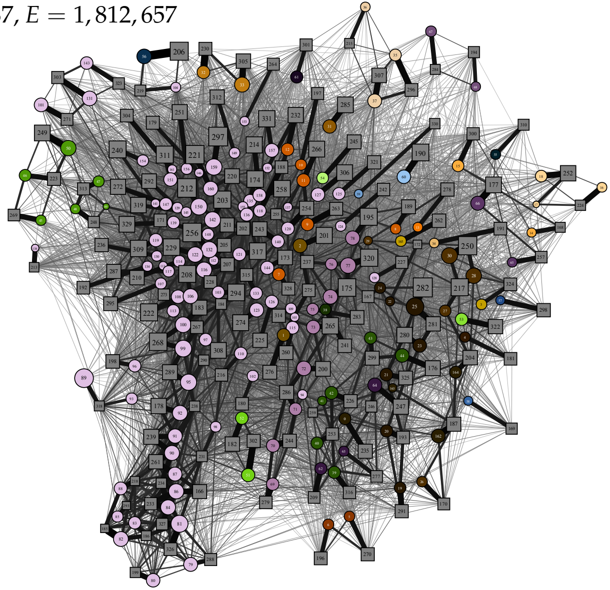

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- Temporal
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- Type/Genre

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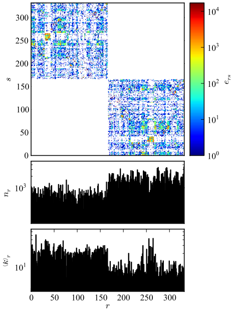

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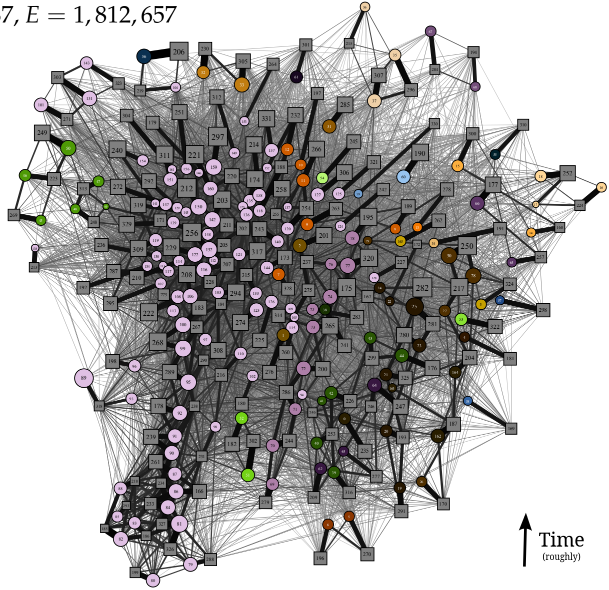

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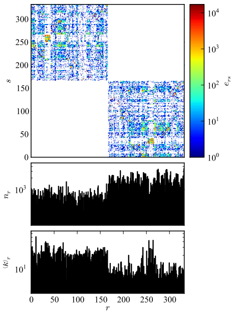

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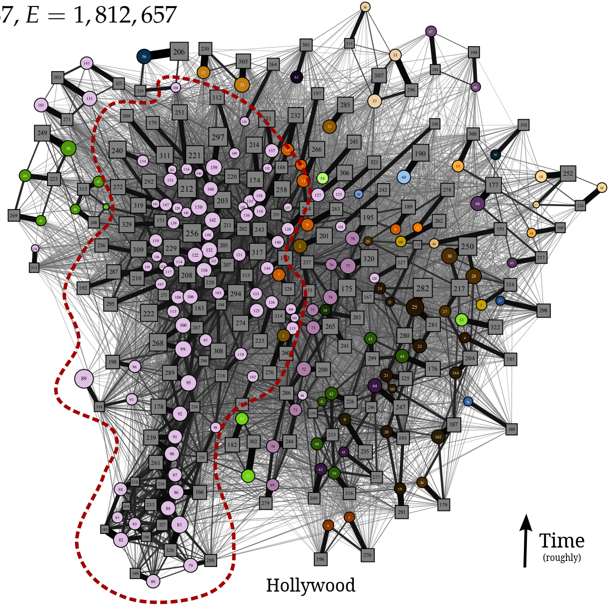

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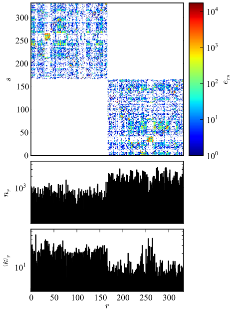

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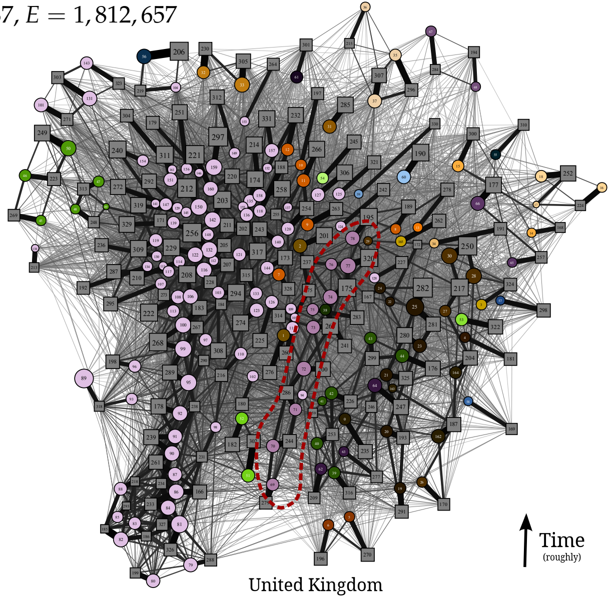

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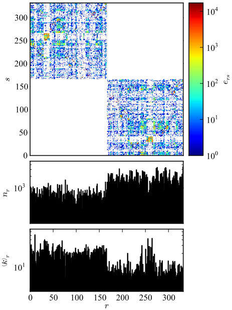

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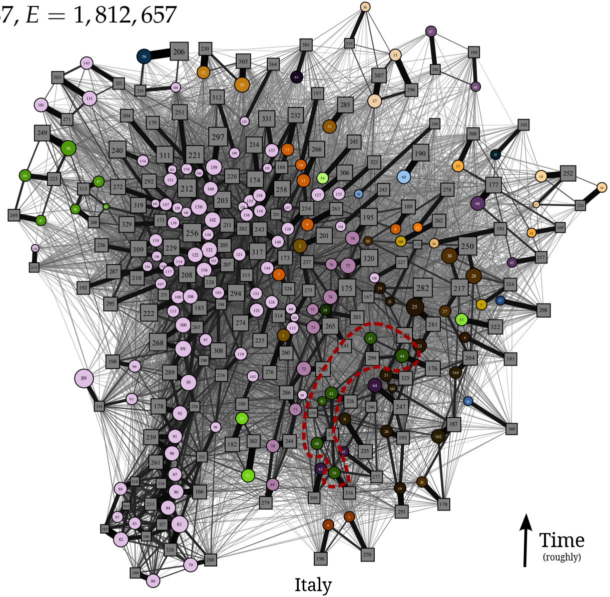

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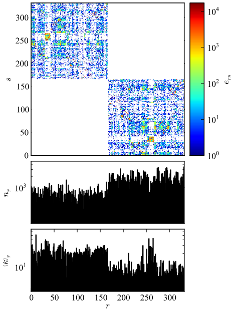

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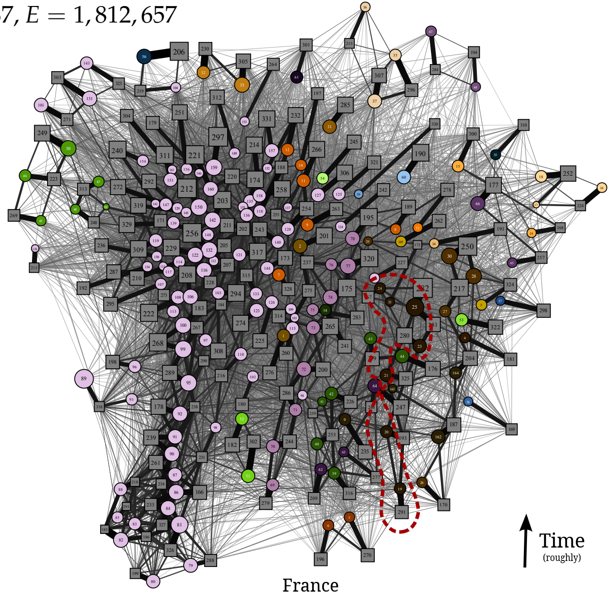

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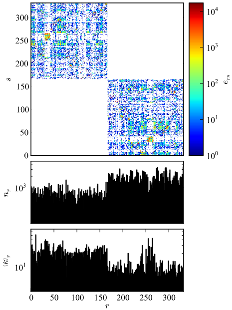

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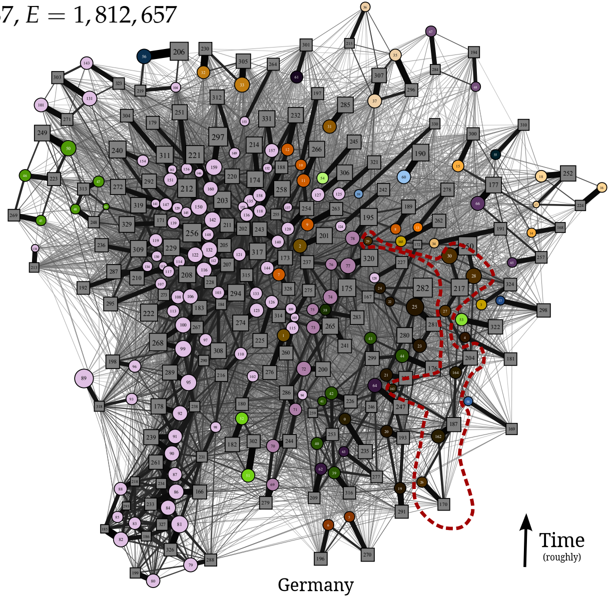
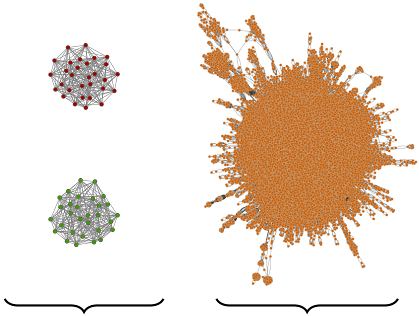


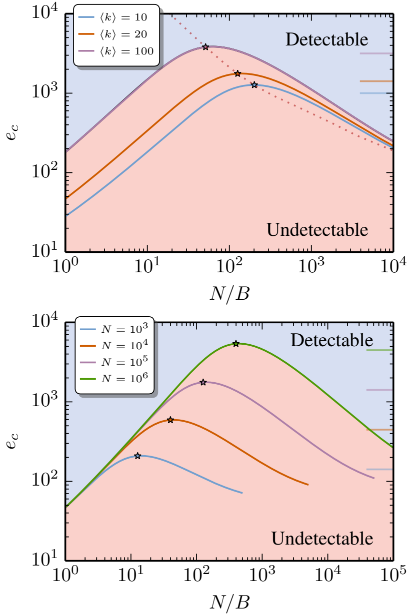
T.P.P., Phys. Rev. Lett. 110, 148701 (2013)

Problem: Resolution limit !?



Minimum detectable block size $\sim \sqrt{N}$.

Similar to modularity!
What is going on?

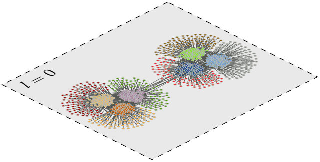


Resolution limit: Lack of prior information

Assumption that all block structures (block graphs) occur with the same probability.

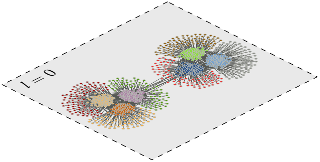
$$\mathcal{L} \sim B^2 \ln E + N \ln B$$

Lack of prior information: Solution → model the model



T.P.P., arXiv:1310.4377

Lack of prior information: Solution → model the model



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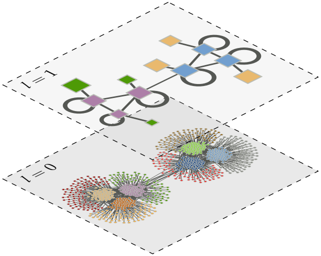

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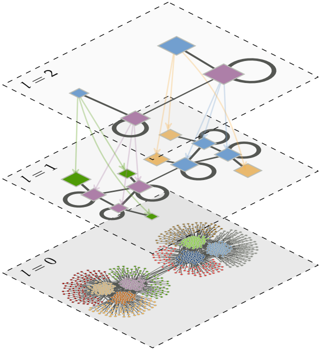

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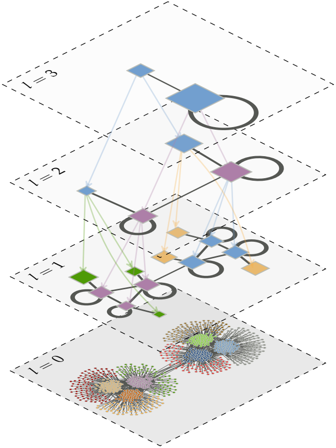
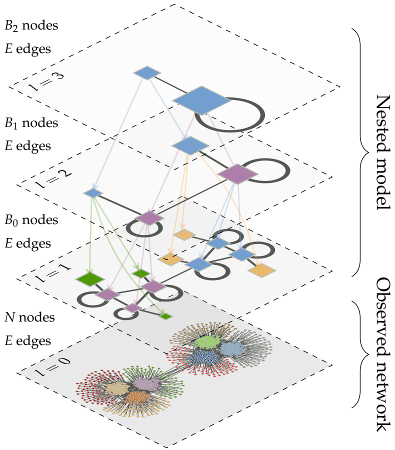


T.P.P., arXiv:1310.4377

LACK OF PRIOR INFORMATION: SOLUTION → MODEL THE MODEL



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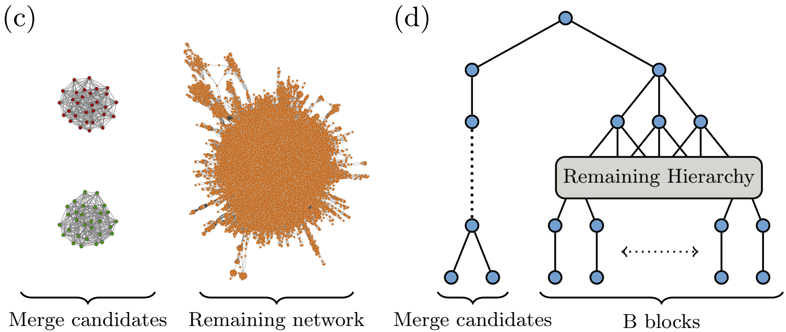
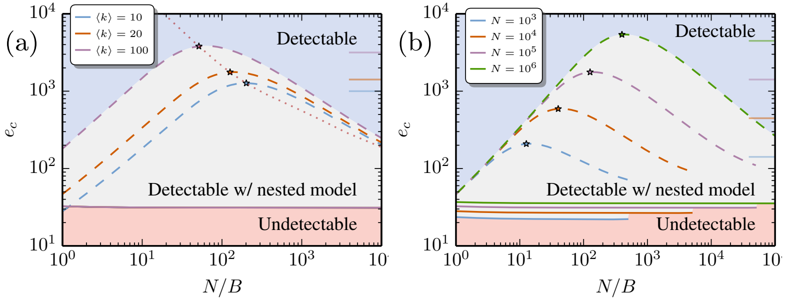
Hierarchical model: Increased resolution



$$e_c^* \approx [\ln(B+N) - \ln n_c] / \ln 2$$

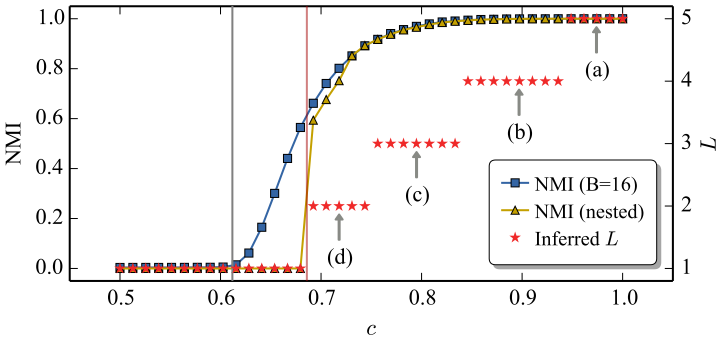
Hierarchical model: Increased resolution

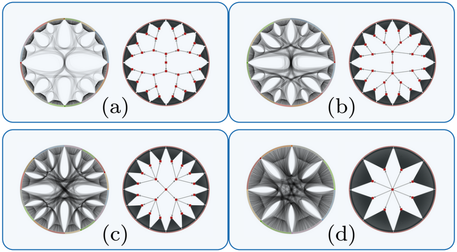


Hierarchical model: $N/B_{\max} \sim \ln N \qquad (\ll \sqrt{N})$

T.P.P., arXiv:1310.4377

Hierarchical model: Built-in validation

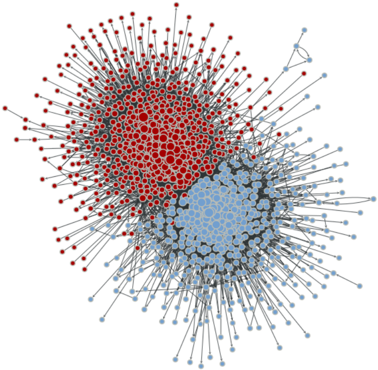




T.P.P., arXiv:1310.4377

EMPIRICAL NETWORKS

POLITICAL BLOGS ($N = 1,222, E = 19,027$)



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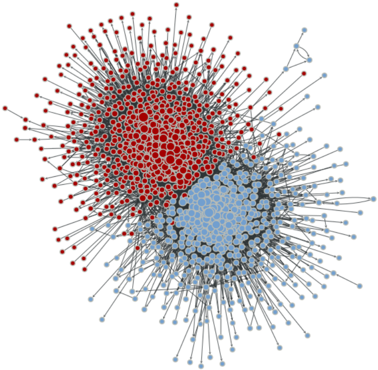
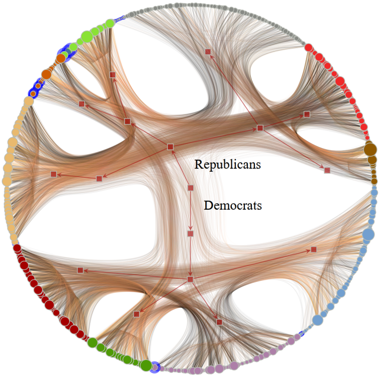
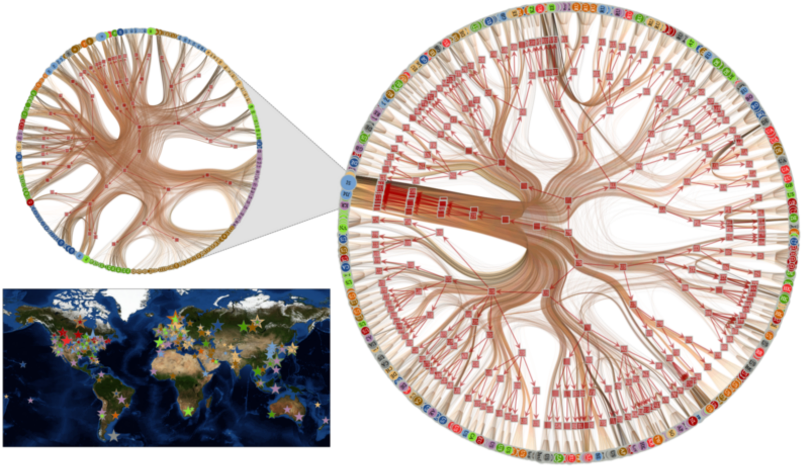




EMPIRICAL NETWORKS

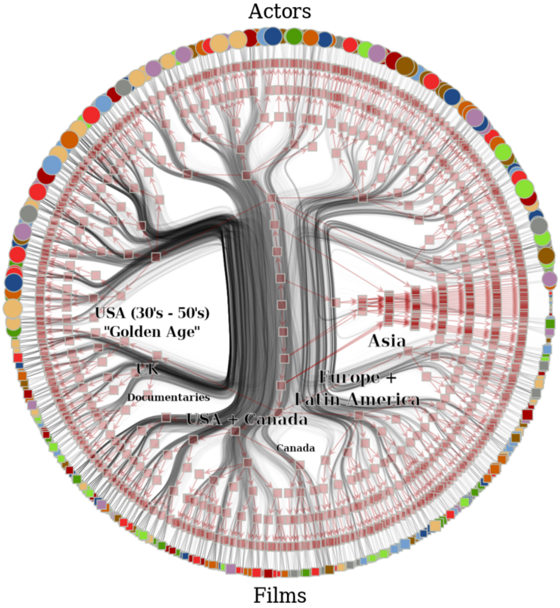
INTERNET (AUTONOMOUS SYSTEMS) ($N = 52,104, E = 399,625$)



T.P.P., arXiv:1310.4377

Empirical networks

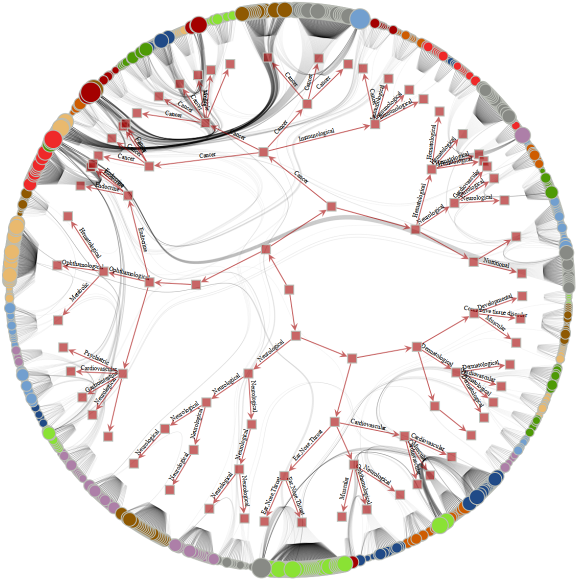
IMDB Film-Actor Network ($N = 372,447, E = 1,812,312, B = 717$)



T.P.P., arXiv:1310.4377

Empirical networks

Human Disease Genes ($N = 903, E = 6{,}760$)



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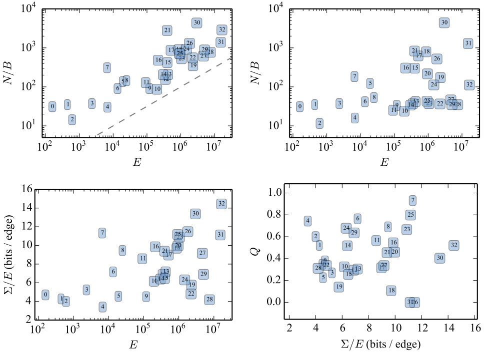

No.	N	E	Dir.	No.	N	E	Dir.	No.	N	E	Dir.
0	62	159	No	11	21,363	91,286	No	22	255,265	2,234,572	Yes
1	105	441	No	12	27,400	352,504	Yes	23	317,080	1,049,866	No
2	115	613	No	13	34,401	421,441	Yes	24	325,729	1,469,679	Yes
3	297	2,345	Yes	14	39,796	301,498	Yes	25	334,863	925,872	No
4	903	6,760	No	15	52,104	399,625	Yes	26	372,547	1,812,312	No
5	1,222	19,021	Yes	16	56,739	212,945	No	27	449,087	4,690,321	Yes
6	4,158	13,422	No	17	75,877	508,836	Yes	28	654,782	7,499,425	Yes
7	4,941	6,594	No	18	82,168	870,161	Yes	29	855,802	5,066,842	Yes
8	8,638	24,806	No	19	105,628	2,299,623	No	30	1,134,890	2,987,624	No
9	11,204	117,619	No	20	196,591	950,327	No	31	1,637,868	15,205,016	No
10	17,903	196,972	No	21	224,832	394,400	Yes	32	3,764,117	16,511,740	Yes

No.	Network	No.	Network	No.	Network
0	Dolphins	11	arXiv Co-Authors (cond-mat)	22	Web graph of `stanford.edu`.
1	Political Books	12	arXiv Citations (hep-th)	23	DBLP collaboration
2	American Football	13	arXiv Citations (hep-ph)	24	WWW
3	C. Elegans Neurons	14	PGP	25	Amazon product network
4	Disease Genes	15	Internet AS (Caida)	26	IMDB film-actor (bipartite)
5	Political Blogs	16	Brightkite social network	27	APS citations
6	arXiv Co-Authors (gr-qc)	17	Epinions.com trust network	28	Berkeley/Stanford web graph
7	Power Grid	18	Slashdot	29	Google web graph
8	arXiv Co-Authors (hep-th)	19	Flickr	30	Youtube social network
9	arXiv Co-Authors (hep-ph)	20	Gowalla social network	31	Yahoo groups (bipartite)
10	arXiv Co-Authors (astro-ph)	21	EU email	32	US patent citations

Conclusion

- Statistical inference → more principled
- Stochastic blockmodels → simple, tractable, **general block structure!**
- Minimum description length (MDL) → non-parametric, elegant, principled
- MDL → **Built-in validation!**
- Nested-stochastic blockmodel → block structure at multiple scales
- Nested MDL → **Vanishing resolution limit!**
- Overall very efficient → meaningful results from very large data sets.

graph-tool

Very fast, freely available C++ code as part of the graph-tool Python library!
`http://graph-tool.skewed.de`