The Manufacturing of Randomness

Art or Science?

Stephan Mertens

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In reality:

- `rand()` returns a pseudo random number

Any one who considers arithmetic methods of producing random digits is, of course, in a state of sin.
John von Neumann (1953)

The practioner's attitude

- Quality: "This RNG has passed a battery of tests."
- Tradition: "I got this RNG from my PhD advisor."
- Ignorance: "RNG? Boring detail."

Consequences:

- Type of underlying RNG seldom published.
- Simulations rarely cross-checked with different RNGs
- Wrong results remain unnoticed. . .

The Ferrenberg Affair

- "high quality" RNGs do fail in stat. mech. simulation†
- triggered a lot of empirical research
- "physical simulations" are added to "battery of tests"

After 40 years of development, one might think that the making of random numbers would be a mature and trouble-free technology, but it seems the creation of unpredictability is ever unpredictable.
Brian Hayes (1993)

† Ferrenberg, Landau, Wong *PRL* **69** 3382 (1992)

Randomness as a Resource

1 rand. bit/s

Prob. (same side up) $\simeq 51\%^\dagger$

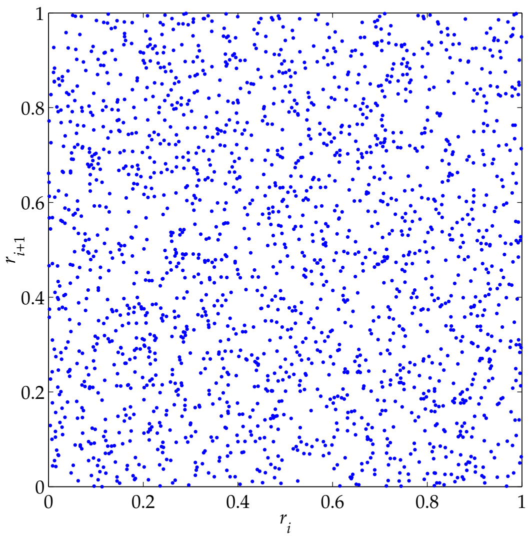
10^{12} random bits/s*

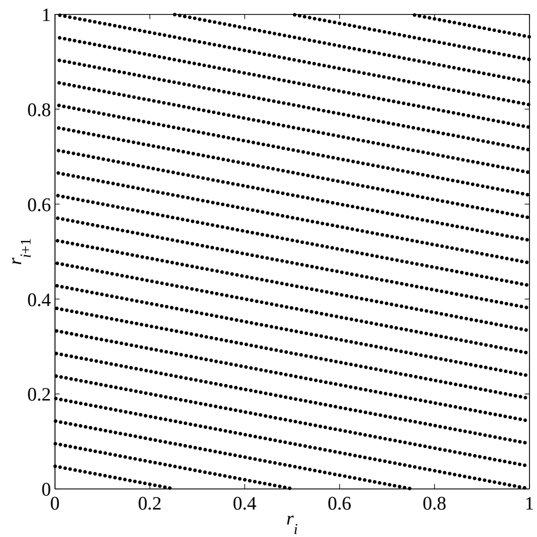
quality is a moving target

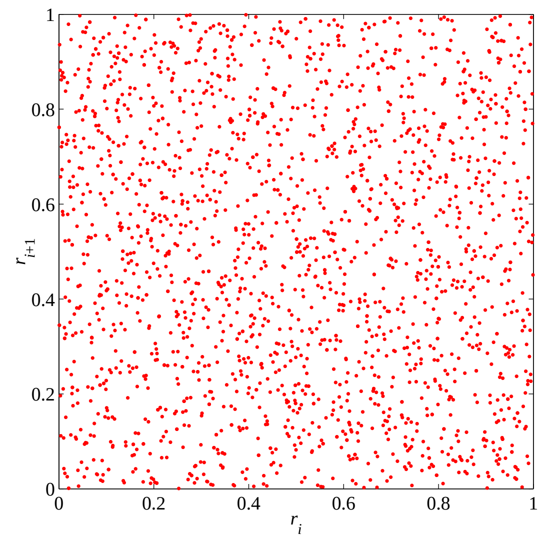
† Persi Diaconis, AAAS Conference, Seattle 2004

* http://tina.nat.uni-magdeburg.de

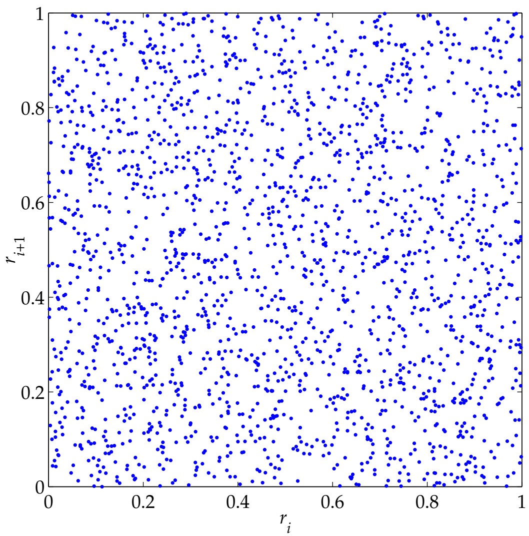
Example



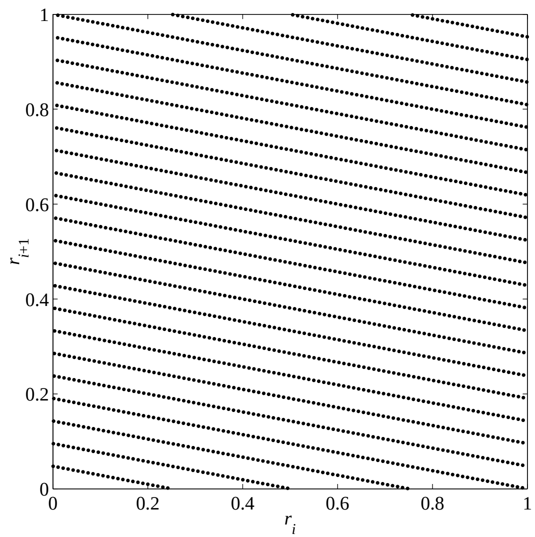




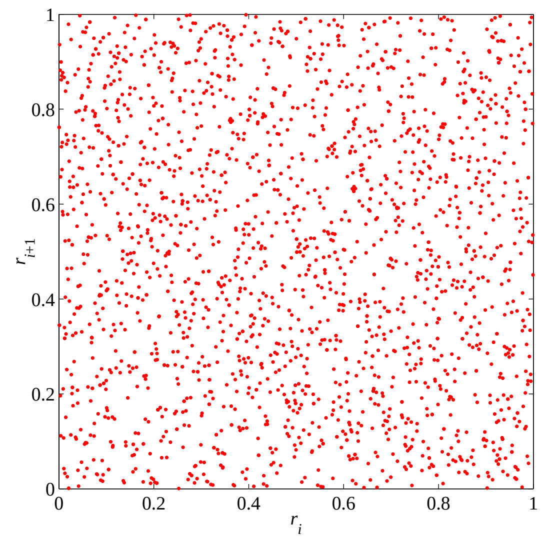
Example



`/dev/random`

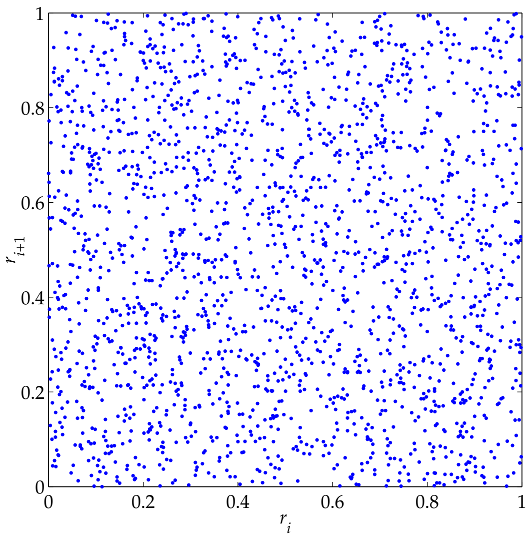


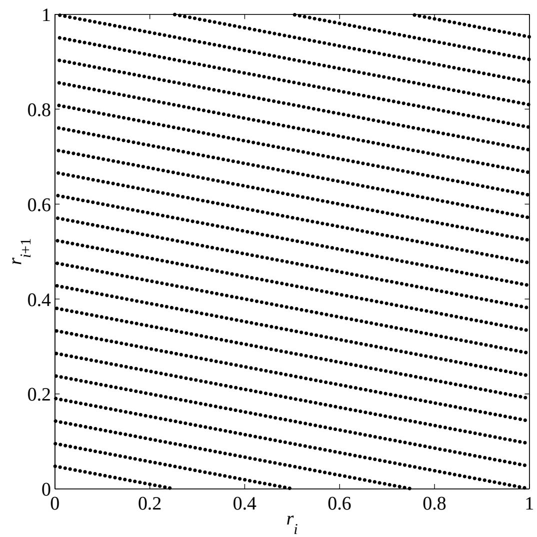
$$r_{i+1} = 95 r_i \bmod 1999$$

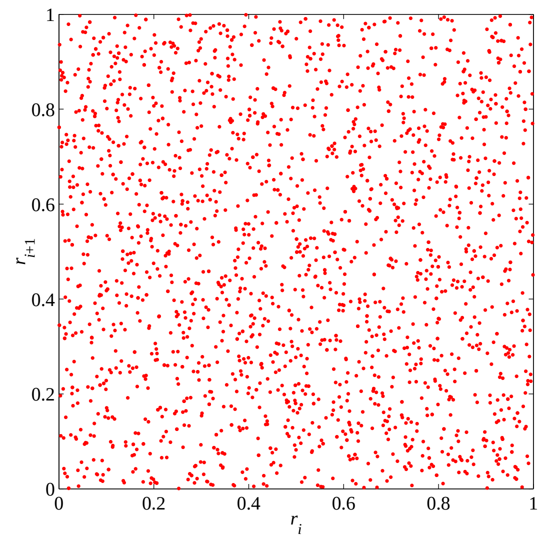


$$x_{i+1} = 95 x_i \bmod 1999$$
$$r_i = 1099^{x_i} \bmod 1999$$

Example







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Random numbers should not be generated with a method chosen at random. Some theory should be used.
Donald E. Knuth (1969–1997)

Recursive Randomness

General scheme:

$$x_k = f(x_{k-1}, x_{k-2}, \ldots, x_{k-p})$$

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Linear Feedback Shift Register sequences

LFSR Sequences

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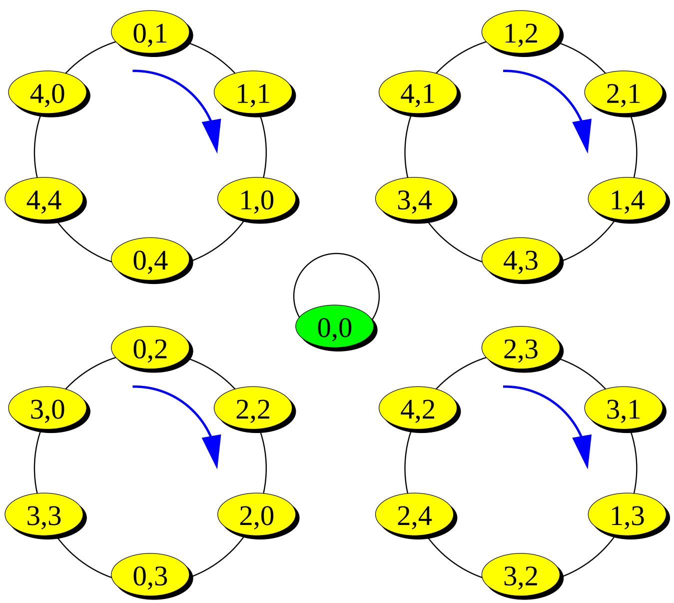
- pseudonoise sequences

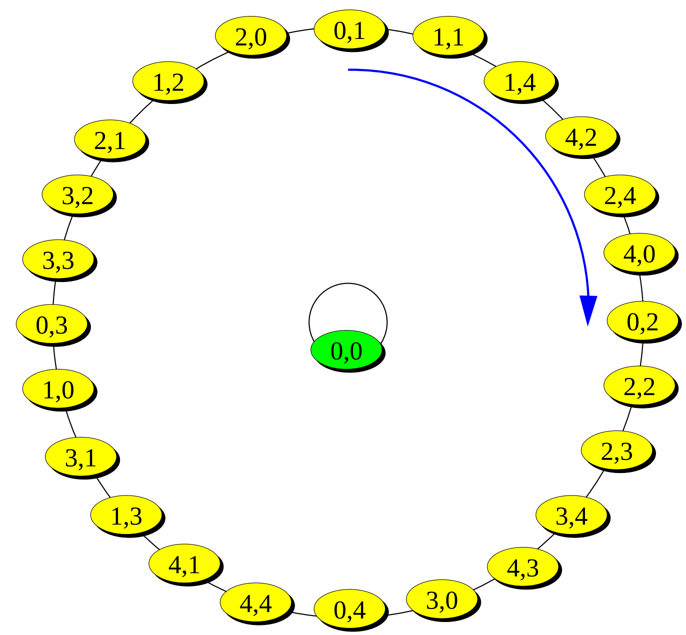
Two Examples in $\mathbb{Z}_5$

$x_k = x_{k-1} + 4x_{k-2} \bmod 5$



$x^2 - x - 4$ (not primitive)

$x_k = x_{k-1} + 3x_{k-2} \bmod 5$



$x^2 - x - 3$ (primitive).

Pseudo Coin Tossing

$$x_k = x_{k-p} + x_{k-q} + \cdots + x_{k-r} \mathrm{mod} 2$$

pseudo noise sequence over $\mathbb{Z}_2$

aka $\mathrm{R}(p, q, \ldots, r)$

Pseudo Coin Tossing

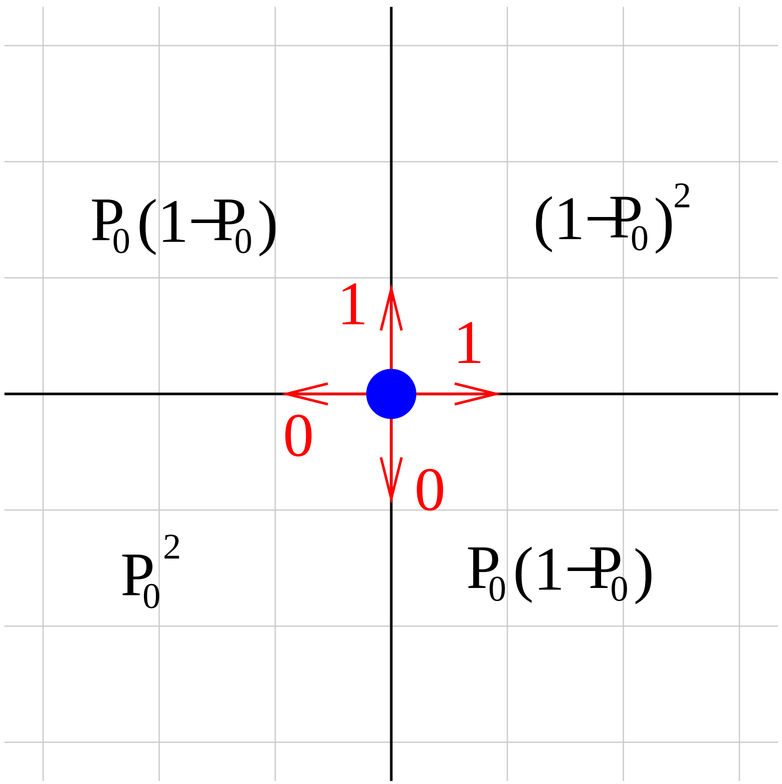
$$x_k = x_{k-p} + x_{k-q} + \cdots + x_{k-r} \bmod 2$$

pseudo noise sequence over $\mathbb{Z}_2$

aka $\mathrm{R}(p, q, \ldots, r)$

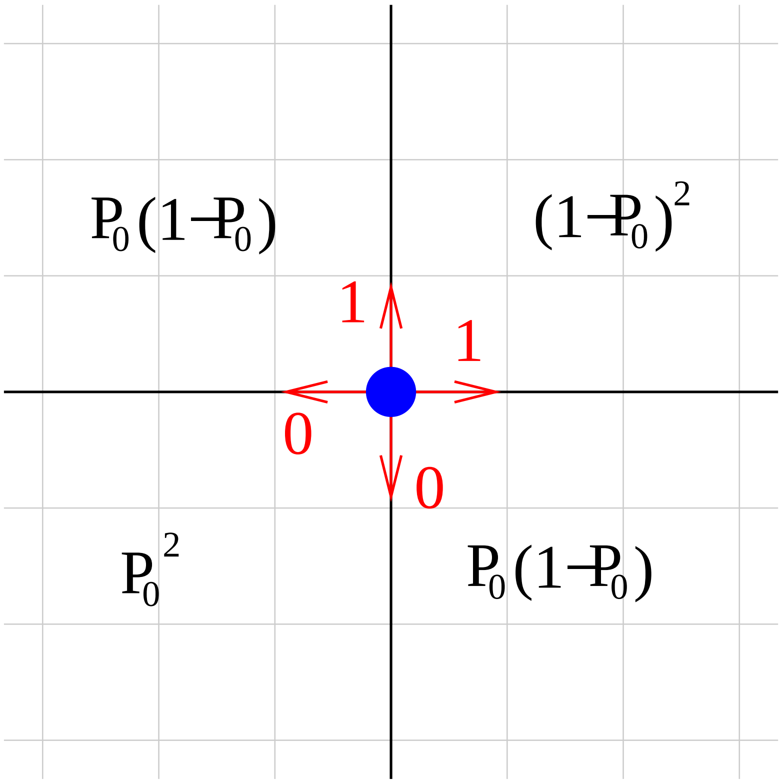
- proposed in 1959 (Green, Smith, Klem)
- popular in physics since 1981 (Kirkpatrick, Stoll)
- "well known instances":
  $\mathrm{R}(250, 103)$, $\mathrm{R}(607, 273)$, $\mathrm{R}(1279, 418)$.

Tracing the Random Walker



$P_0(w)$: Prob. of having more O's than 1's in w-tuple

Tracing the Random Walker



Vattulainen et. al ('94):

$\mathrm{R}(250, 103)$ bad

$\mathrm{R}(250, 201, 152, 103)$ O.K.

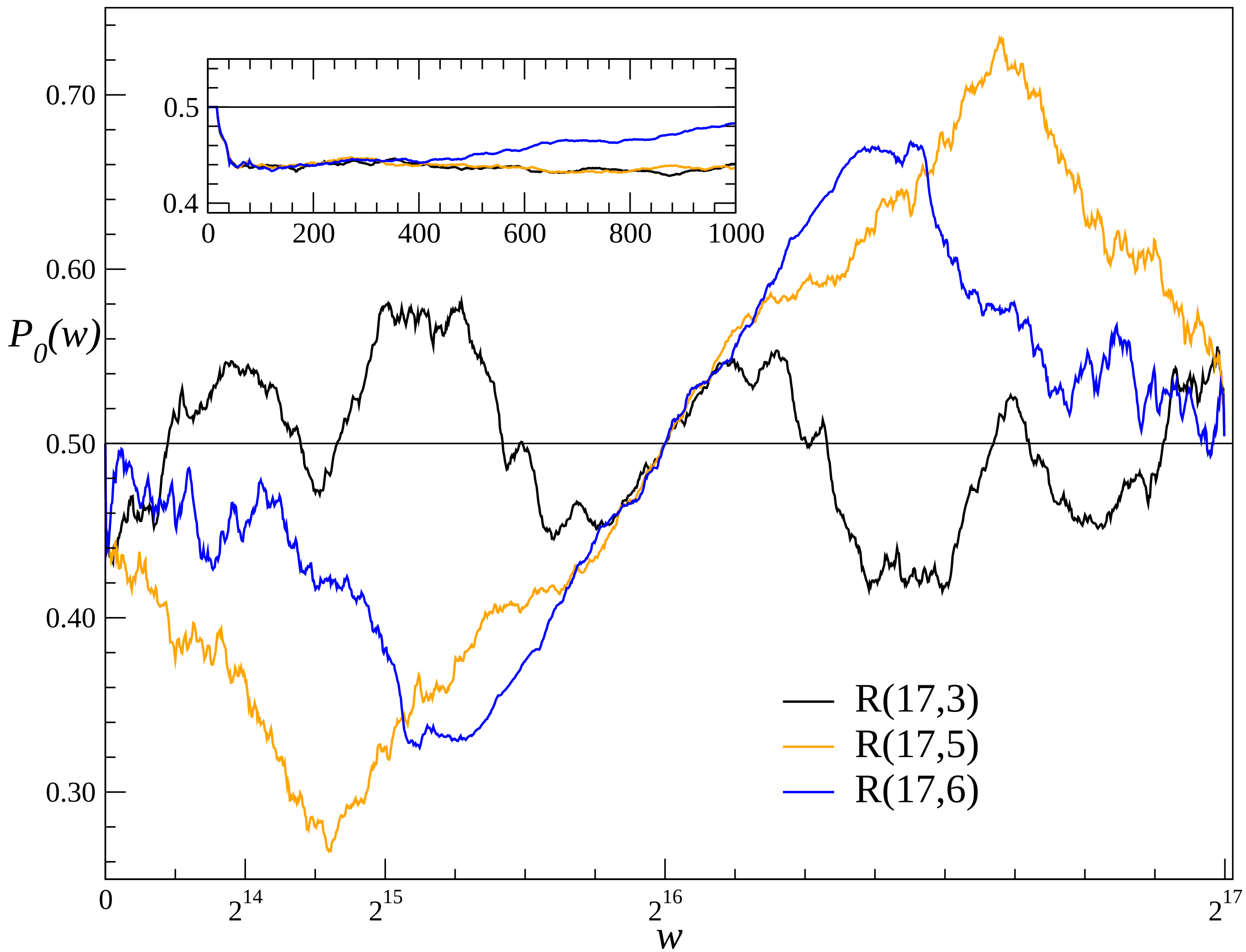
Ziff ('98):

$\mathrm{R}(9689, 471)$ bad

$\mathrm{R}(9689, 6988, 1586, 471)$ O.K.

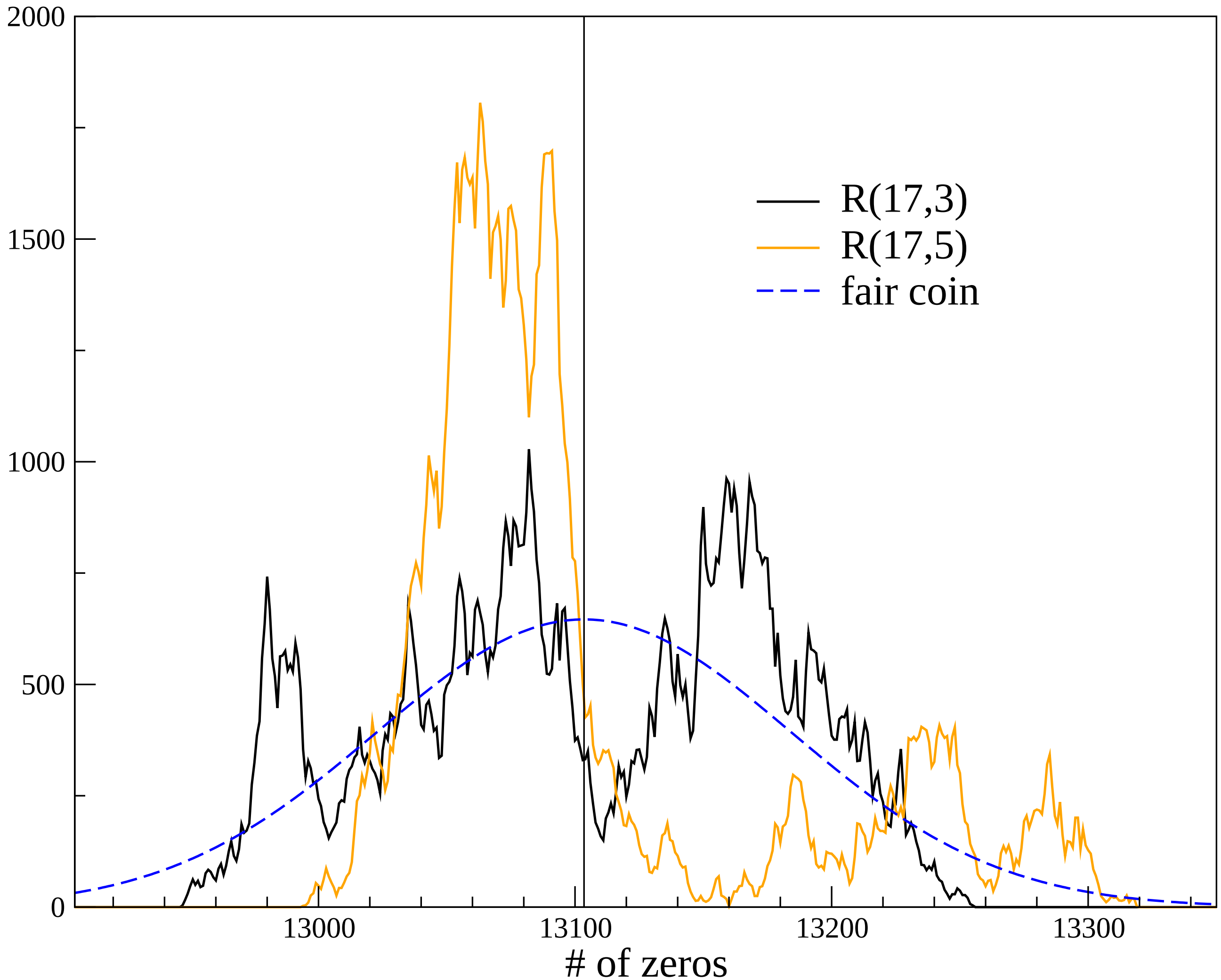
$P_0(w)$: Prob. of having more O's than 1's in w-tuple

Heads vs. Tails



Probability to have a majority of 0's in tuples of size w.

Heads vs. Tails



Tuples of size $w = 26207$.

Calculating $P_0(w)$

Generating function: $f_w(z) = \sum_{n=0}^{w} p_1(w,n)\, z^n$

|

Prob. of having n 1's in w-tuple

Calculating $P_0(w)$

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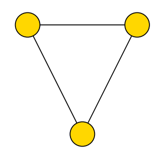
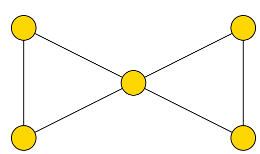
Prob. of having n 1's in w-tuple

$$\frac{1+z}{2}$$



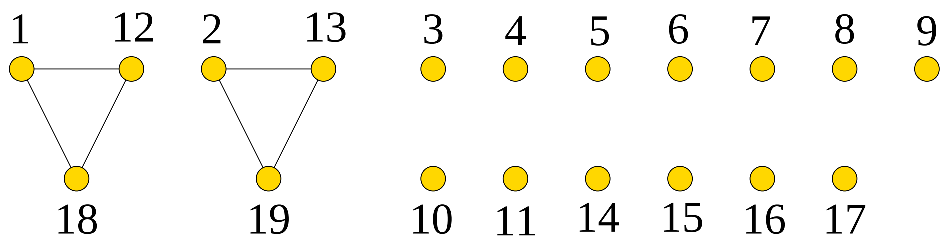
$$\frac{1+3z^2}{4}$$



$$\frac{1+2z^2+4z^3+z^4}{8}$$

Calculating $P_0(w)$

Example: $P_0(19)$ for $\mathrm{R}(17,6)$



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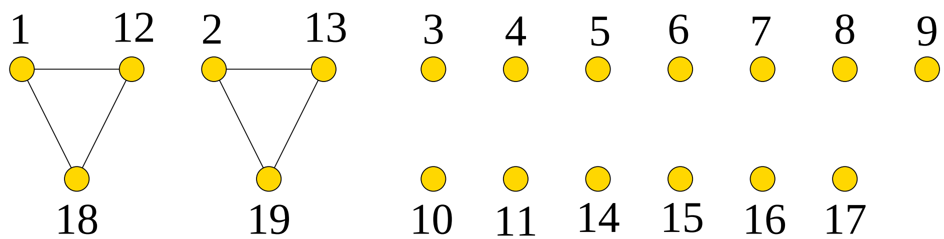

$$f_{19}(z) = \left(\frac{1+z}{2}\right)^{13} \left(\frac{1+3z^2}{4}\right)^2 \qquad P_0(19) = \frac{32\,053}{65\,536} \approx 0.4891$$

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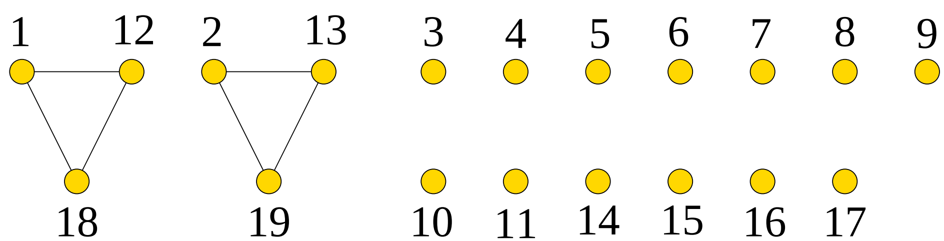

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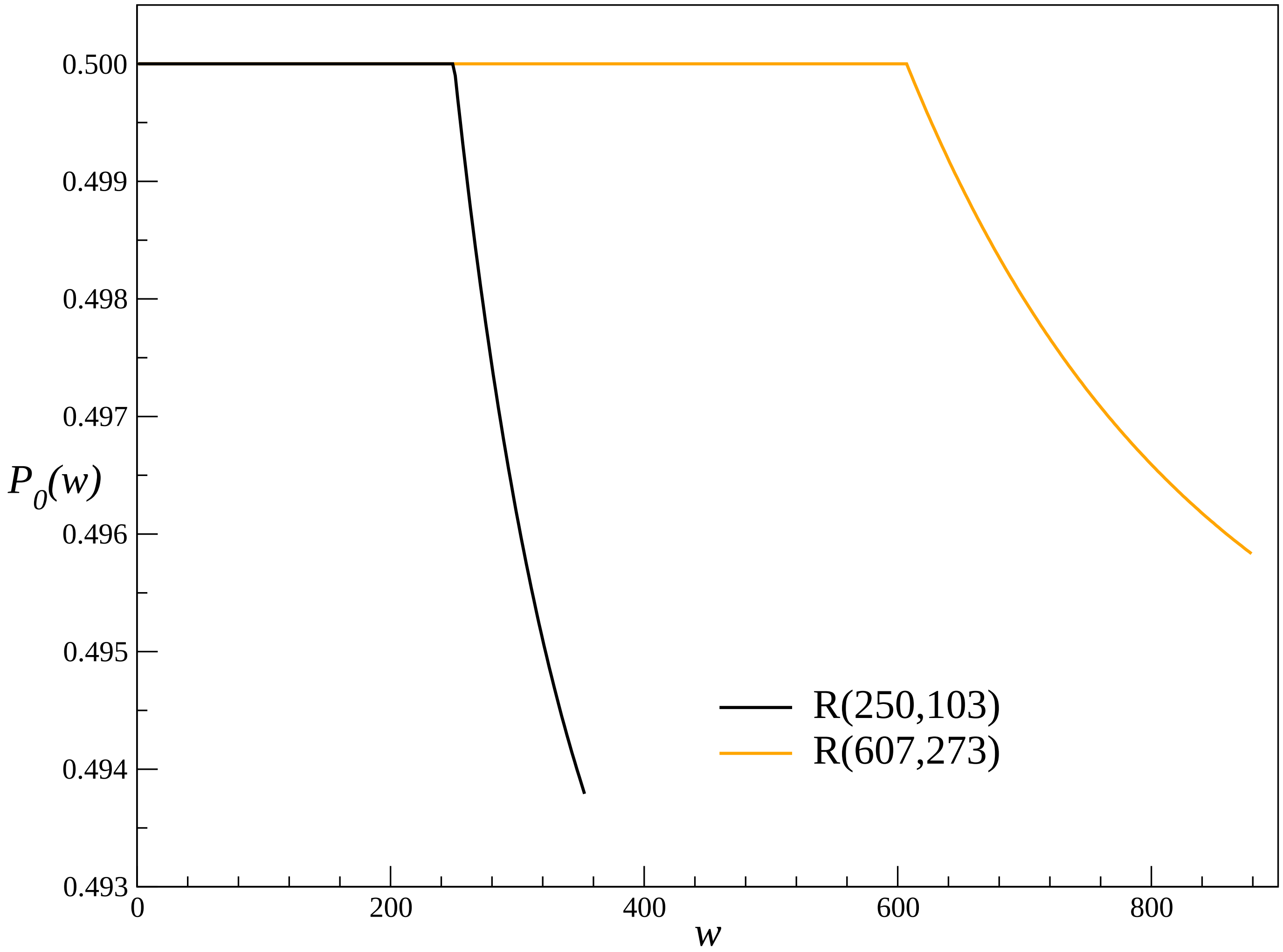
General case: $\mathrm{R}(p,q)$

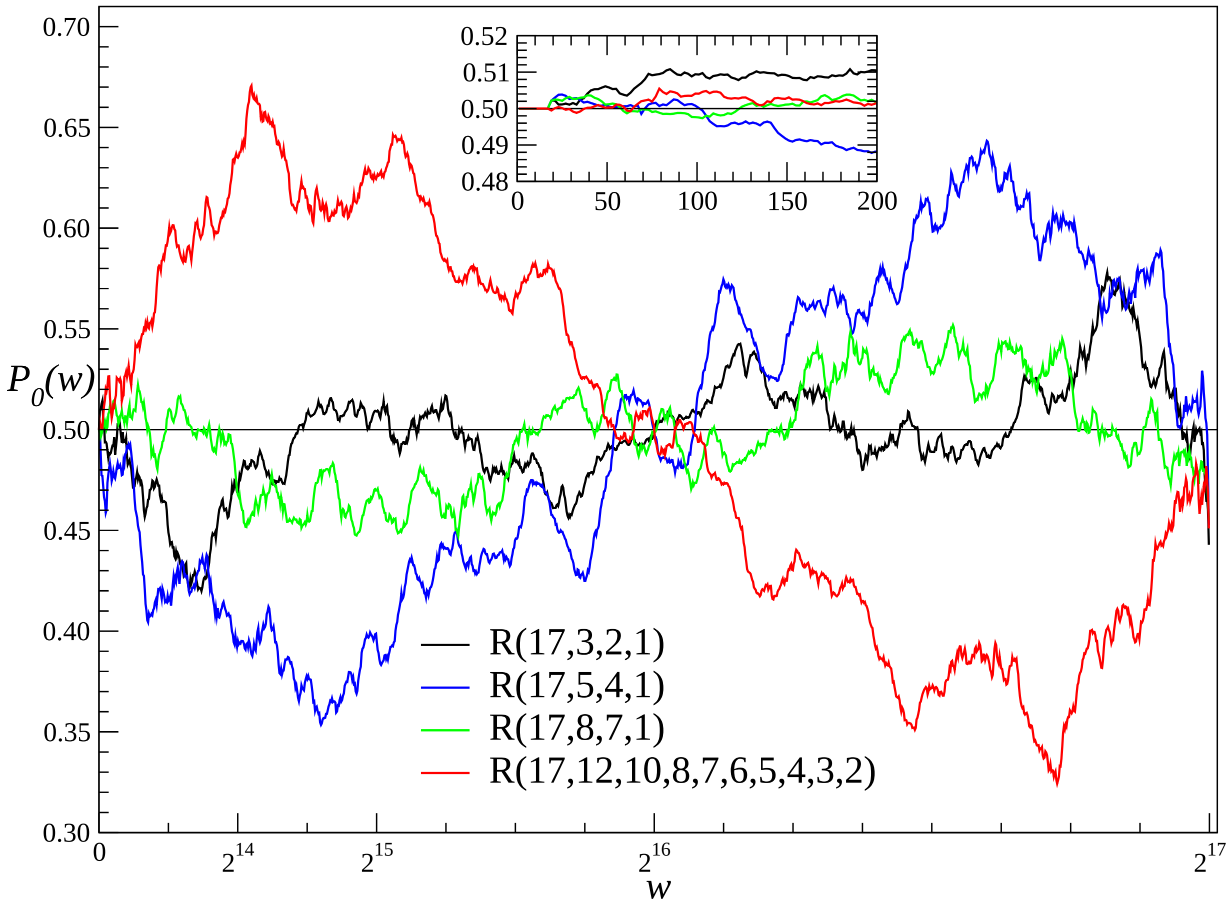
$$P_0(p+1) = \frac{1}{2} - \frac{1}{2^{p+1}(p-1)}\binom{p}{p/2} = \frac{1}{2} - \frac{1}{\sqrt{2\pi p^3}} + \mathcal{O}(p^{-5/2})$$

Heads vs. Tails



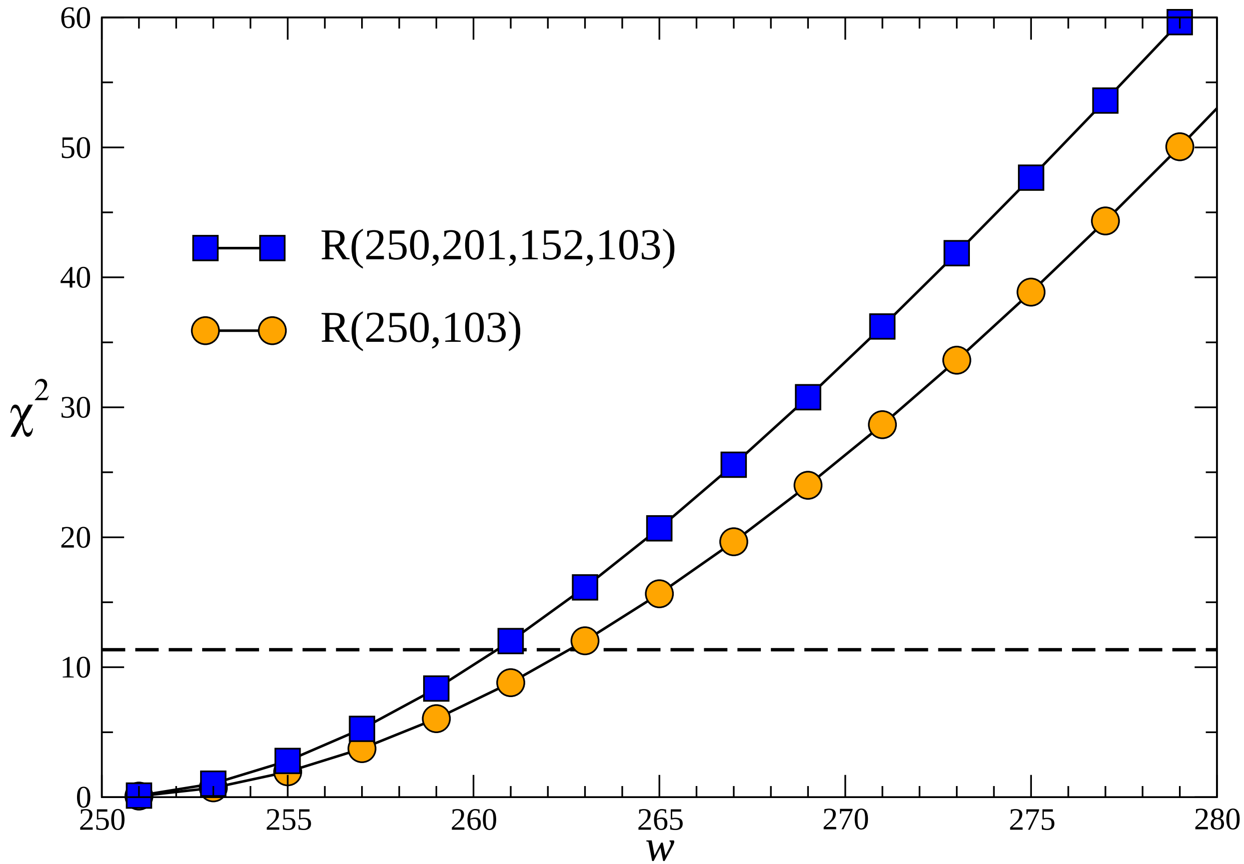
Bias in “industrial sized” random number generators .

More Feedback



$$P_0(p+1) = \frac{1}{2} + (-1)^{t/2} \frac{(t-1)!!}{\sqrt{2\pi p^{t+1}}} + \mathcal{O}(p^{-\frac{t+3}{2}})$$

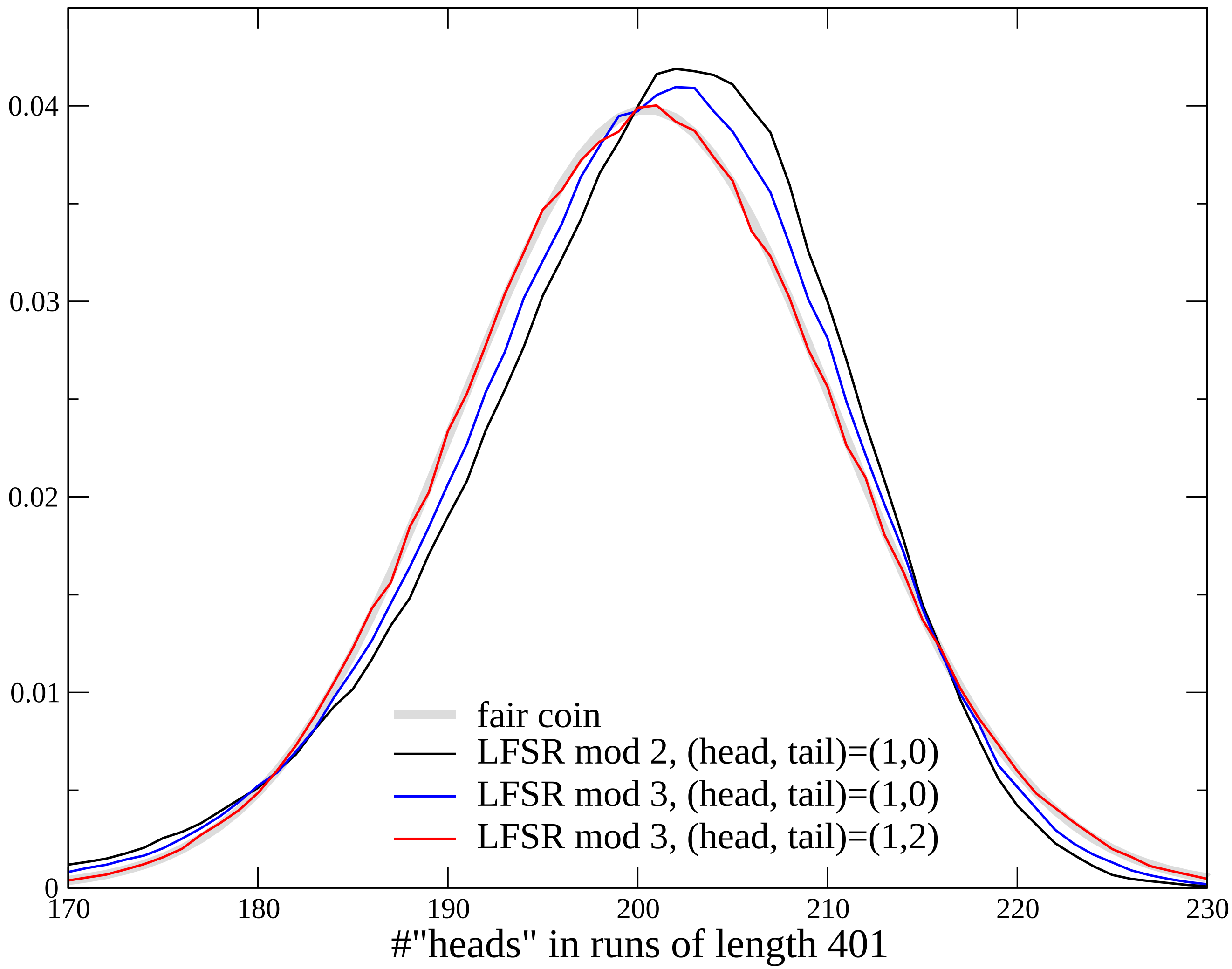
Bias in Random Walks



10^6 samples for $\mathrm{R}(250, 103)$, 10^{10} for $\mathrm{R}(250, 201, 152, 103)$.

empirical quality expires

Avoid the Zeros



The Ferrenberg Drama

Protagonists: Lagged Fibonacci RNGs $F(p, q, \circ)$,

$$x_k = x_{k-q} \circ x_{k-p} \bmod m$$

with properly chosen magic numbers p and q and

$$\circ \in \{\oplus, +, -, \times\}.$$

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The plot: cluster Monte Carlo simulations in Ising systems.

- $F(p, q, \times)$ o.k.
- $F(p, q, \pm)$ bad
- $F(p, q, \oplus)$ worse

The Wolff Algorithm

Initialize:

- Put a randomly chosen spin in the cluster.
- Put its equally aligned neighbours in the candidate list.

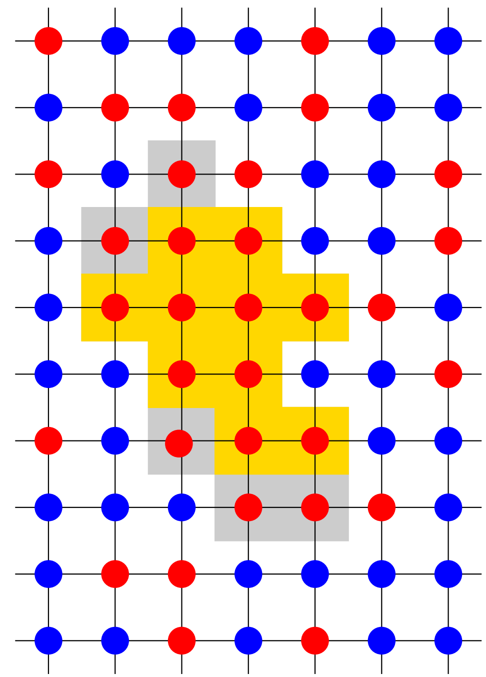
Grow cluster:

While candidate list of spins is not empty

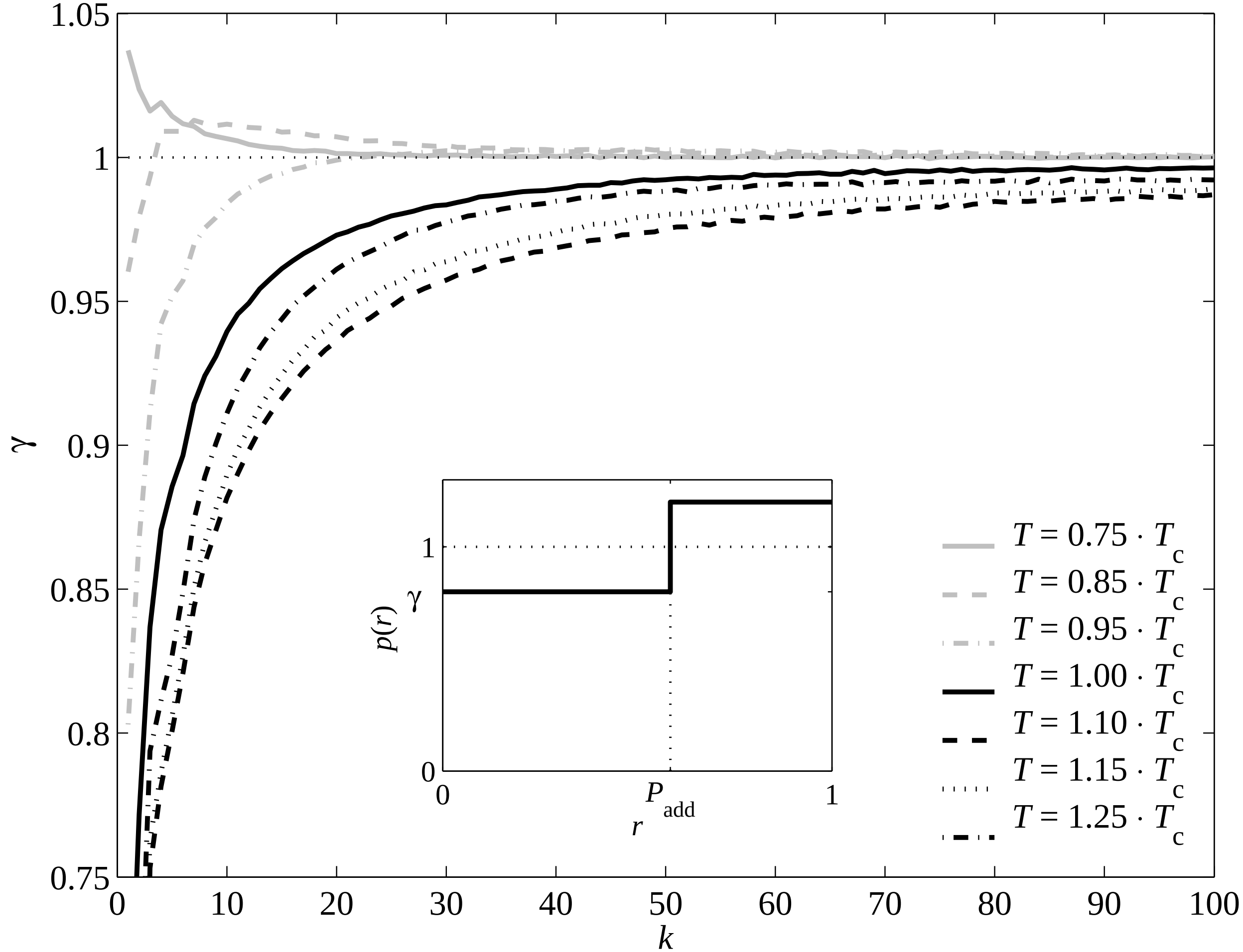
1. remove a spin from the candidate list
2. with probability $P_{\mathrm{add}} = 1 - e^{-2/T}$:
   - add this spin to the cluster
   - add its equally aligned neighbors to the candidate list if they are not in the cluster

Flip all spins in the cluster.

Repeat

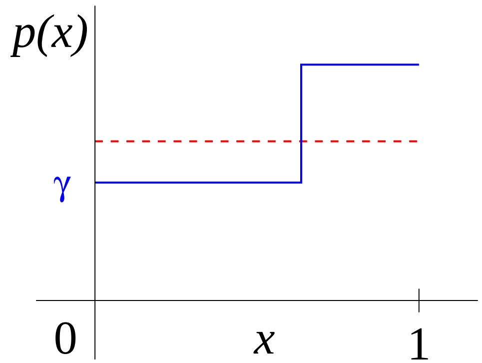


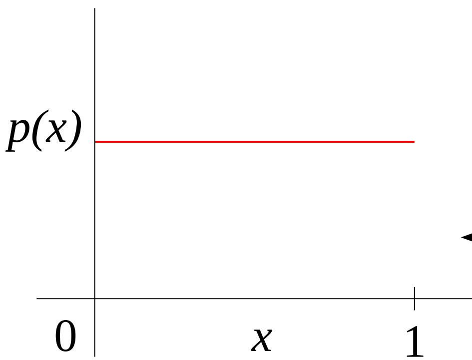
Algorithmically Induced Bias



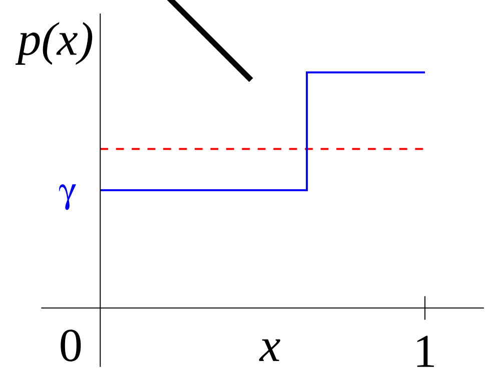
Wolff algorithm on a 24×24 spin Ising model.

Restoring the Flat Measure

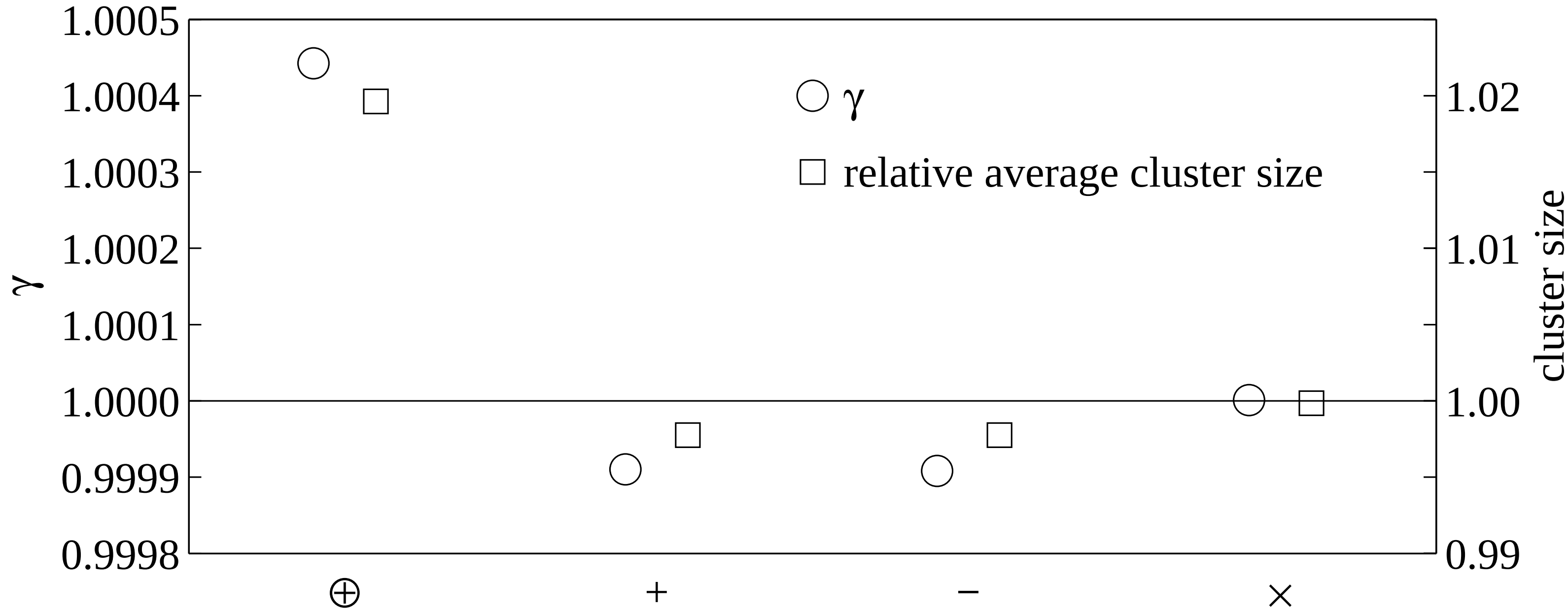




$$x_k = f(x_{k-1}, x_{k-2}, \ldots, x_{k-p})$$



Bias and Clustersize



$$\gamma_{\text{in}} = 0.975$$

Simulation: $F(13, 33, \circ)$ on a 16×16 spin system.

A Model RNG

$$r_k = \alpha(r_{k-1} + r_{k-2} + \cdots + r_{k-p}) \bmod 1 \qquad r_i \in [0,1)$$

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$\rho_k(r)$: pdf. of r_k:

$$\rho_k(r) = \frac{1}{\alpha}\sum_{j=0}^{\lfloor p\alpha \rfloor} \rho_{k-1} \star \rho_{k-2} \star \cdots \star \rho_{k-p}\left(\frac{r+j}{\alpha}\right)$$

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This is the Riemann sum approximation of the integral

$$\int \rho_{k-1} \star \cdots \star \rho_{k-p}(x)\,\mathrm{d}x = 1$$

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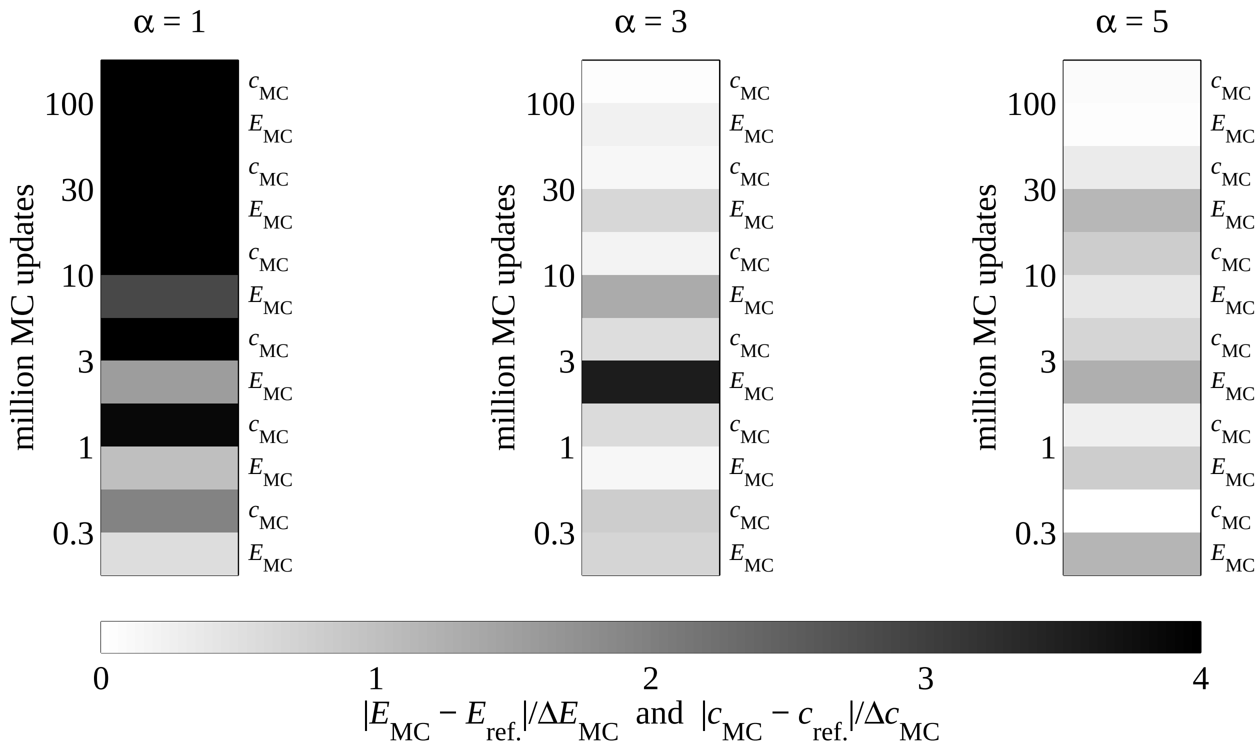
$$\int \rho_{k-1} \star \cdots \star \rho_{k-p}(x)\,\mathrm{d}x = 1$$

$\lim_{\alpha\to\infty} \rho_k(r) = 1$ independently of $\rho_{k-1}, \ldots, \rho_{k-p}$

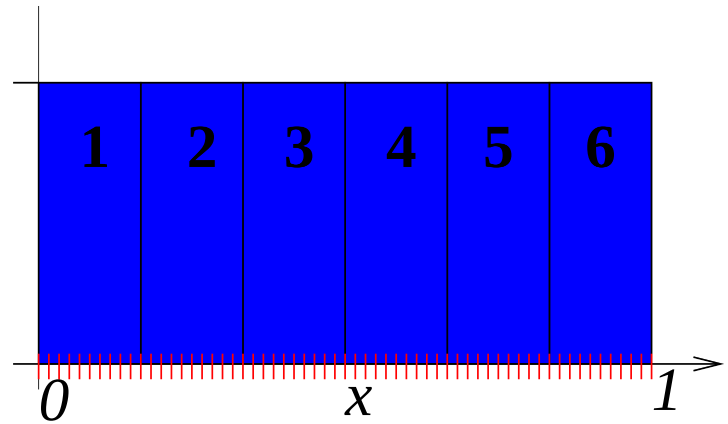
Curing the Ferrenberg Syndrome



Wolff algorithm on $6 \times 6 \times 6$ Ising model

RNG: $x_k = \alpha(x_{k-13} + x_{k-33}) \bmod (2^{31} - 1)$

Macrostate Entropy



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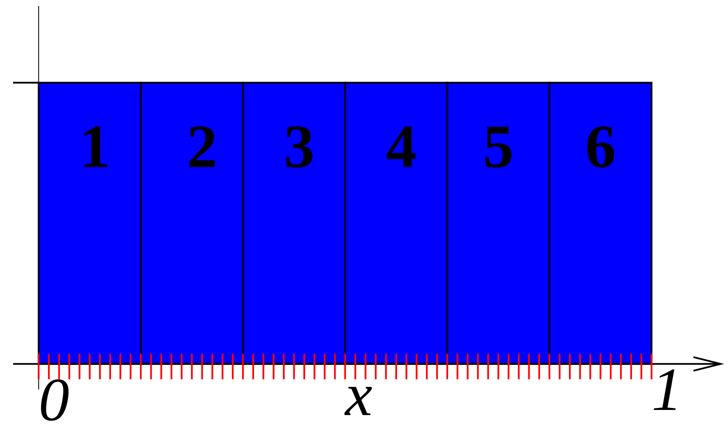
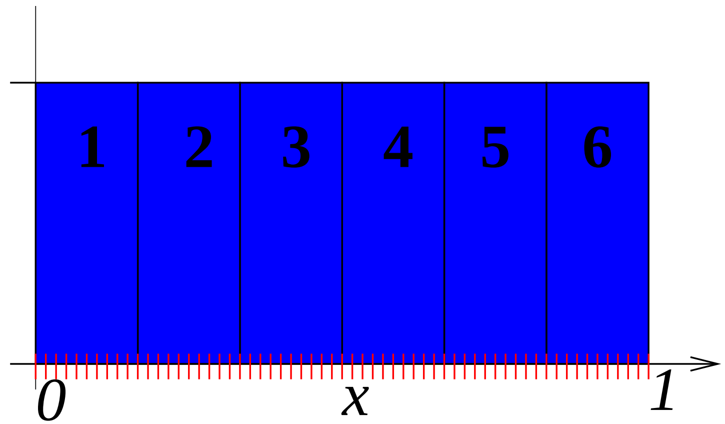


2^{32} microstates $\quad x_{n+1} = f(x_1, \ldots, x_n) \quad$ deterministic

M macrostates $\quad (m_1, \ldots, m_n) \mapsto m_{n+1} \quad$ random

Macrostate Entropy



2^{32} microstates	$x_{n+1} = f(x_1, \ldots, x_n)$	deterministic
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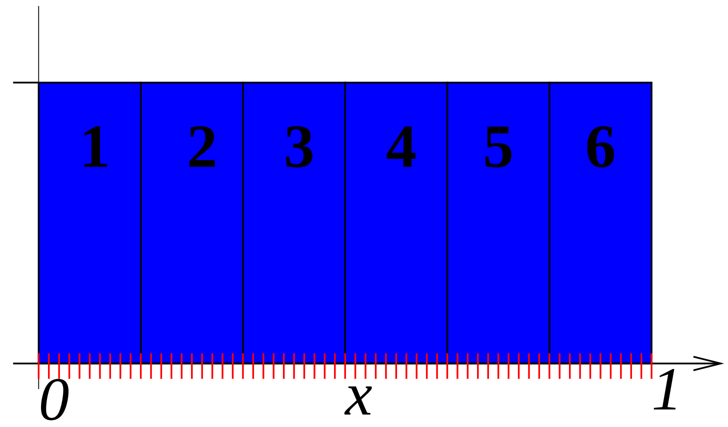
$$H = -\sum_{\{m_i\}} \mathbb{P}(m_1, \ldots, m_{n+1}) \log_2 \frac{\mathbb{P}(m_1, \ldots, m_{n+1})}{\mathbb{P}(m_1, \ldots, m_n)} \leq \log_2 M$$

Macrostate Entropy



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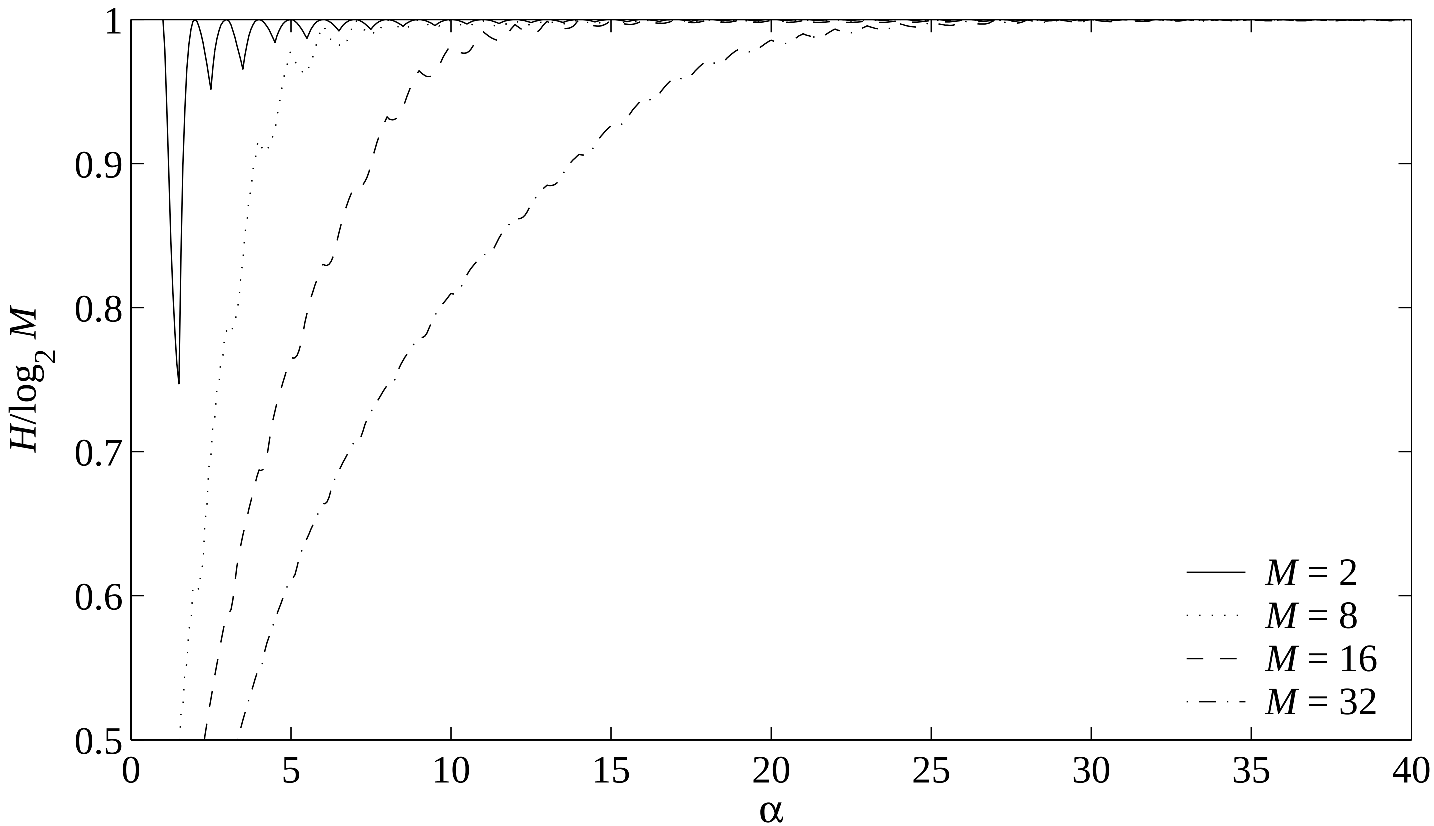
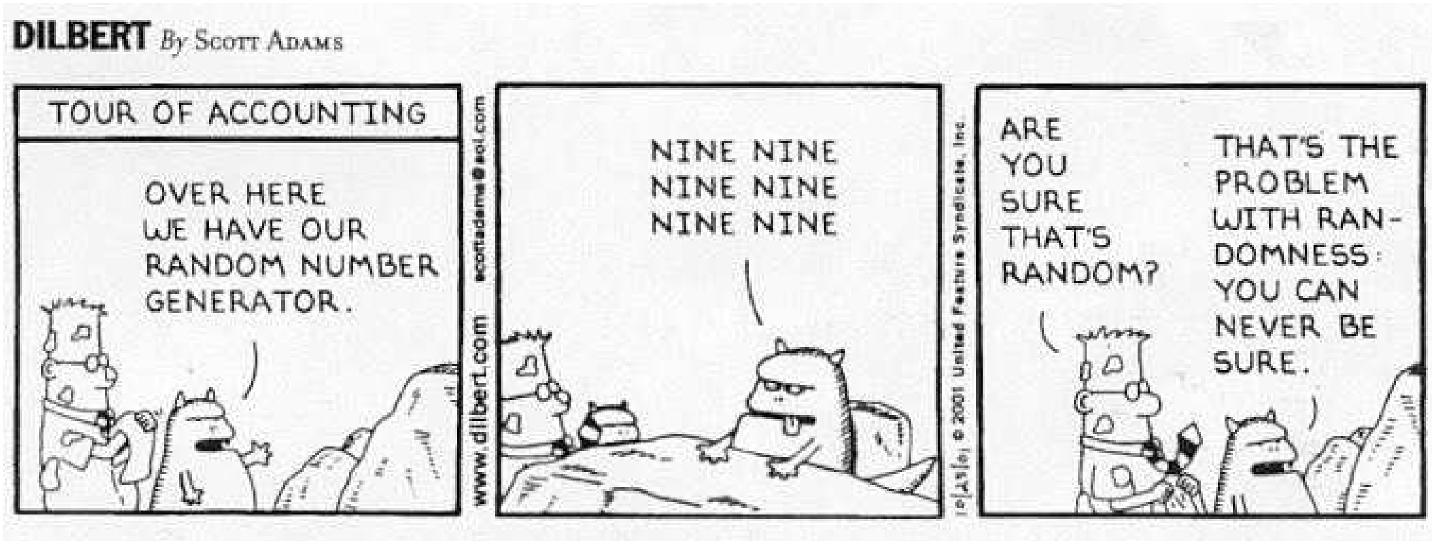
property of production rule f (RNG) and macrostates M (application) only

Macrostate Entropy



$$r_i = \alpha(r_{i-p} + r_{i-q}) \bmod 1$$

Finis



- Pseudo Random Coins Show More Heads Than Tails

  H. Bauke, S.M., *J. Stat. Phys.* **114** 1149-1169 (2004)

- Entropy of Pseudo Random Number Generators

  S.M., H. Bauke, *Phys. Rev. E* **69** 055702(R) (2004)

- `http://www.uni-magdeburg.de/mertens`
- `http://tina.nat.uni-magdeburg.de/intern/software/trng`