

# The Multilevel Monte Carlo Method: basic concepts and further developments

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Oberseminar Computerorientierte Theoretische Physik

20 December 2017

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**Question:** Suppose  $X$  is a Random Variable, such that

- $X$  is not available in closed form
- $X$  is available through its i.i.d. samples  $X^i$

How to compute accurately and quickly the mean value

$$\mu = \mathbb{E}[X] = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) \quad ?$$

**Monte Carlo:** Use sample average:

$$E_M[X] := \frac{1}{M} \sum_{i=1}^M X^i.$$

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$$\text{MSE} = \mathbb{E}[Z^2], \quad Z = E_M[X] - \mu.$$

### Theorem

$$\text{MSE} = \frac{1}{M} \text{Var}[X].$$

### Drawbacks:

- Very slow (Root-MSE  $\sim \frac{1}{\sqrt{M}}$ )
- Usually not realistic: approximate samples of  $X_N \approx X$ .

Here  $N$  is a „discretization parameter“, e.g.

- # particles in a MD-Simulation
- # dof in a Finite Element / Finite Difference approximation
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- In particular, if
  - Approx. error  $\sim N^{-\alpha}$
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# Two-Level Monte Carlo

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*(m < M: faster sampling for the same accuracy)*

Extension to multiple levels

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**Multilevel Monte Carlo**

$$E^{ML}[X] := \sum_{\ell=1}^L E_{M_\ell}[X_\ell - X_{\ell-1}], \quad X_0 = 0.$$

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## Main idea: Equidistrib. of the comput. cost over FE levels

1. HEINRICH, J. Complexity (1998)
2. GILES, Oper. Res. (2008)
3. BARTH, SCHWAB, ZOLLINGER, Numer. Math. (2011)
4. CLIFFE, GILES, SCHEICHL, TECKENTRUP, Comput. Vis. Sci. (2011)
- ...

## Our work [BIERIG/CHERNOV'15+]:

- Multilevel MC approx. of the variance and higher order moments

$$\mu^k = \mathbb{E}(X - \mathbb{E}[X])^k = \int_{-\infty}^{\infty} (x - \mu)^k f_X(x) dx,$$

- Approximation of Probability Density Functions  $f_X$  via Max. Entropy Method

$$f_X \approx \operatorname{argmin} \left\{ \int \rho \ln \rho : \mu^k = \int (x - \mu)^k \rho(x) dx \right\}$$

- Application to the contact with rough random obstacles.

## Multilevel Monte Carlo sample mean estimator:

$$\mathbb{E}[X] \approx E^{ML}[X] = \sum_{\ell=1}^L E_{M_\ell}[X_\ell - X_{\ell-1}], \quad X_0 = 0.$$

Theorem (Accuracy / Cost relation, simplified)

Assume that

a)  $|\mathbb{E}[X - X_\ell]| \lesssim N_\ell^{-\alpha}$ ,   b)  $\text{Var}[X_\ell - X_{\ell-1}] \lesssim N_\ell^{-\beta}$ ,   c)  $\text{Cost}(X_\ell) \lesssim N_\ell^\gamma$ ,

then there exist  $M_\ell$ , s.t.  $\text{RMSE}(E_M) < \varepsilon$  and  $\text{RMSE}(E^{ML}) < \varepsilon$

$$\text{Cost}(E_M) \lesssim \varepsilon^{-2-\frac{\gamma}{\alpha}}, \quad \text{Cost}(E^{ML}) \lesssim \varepsilon^{-2-\frac{\gamma}{\alpha} + \frac{\min(2\alpha, \beta, \gamma)}{\alpha}}. \quad (\gamma \neq \beta)$$

## Proof (sketch for the case $2\alpha > \min(\beta, \gamma)$ ):

- MSE( $E^{ML}$ ) =  $|\mathbb{E}[X_L - X]|^2 + \sum_{\ell=1}^L \frac{1}{M_\ell} \text{Var}[X_\ell - X_{\ell-1}] \sim \varepsilon^2$
  - Balancing the summands:  $N_L^{-2\alpha} \sim \varepsilon^2$  and  $\sum_{\ell=1}^L \frac{N_\ell^\beta}{M_\ell} \sim \varepsilon^2$
  - Finding  $M_\ell$ : Minimize Cost( $E^{ML}$ ) under constraints 
- $$\text{Cost}(E^{ML}) \sim \sum_{\ell=1}^L M_\ell \cdot \text{Cost}(X_\ell)$$
- Optimal choice:  $M_\ell \sim N_\ell^{-\frac{\beta+\gamma}{2}} \Rightarrow \text{Cost}(E^{ML}) \sim \sum_{\ell=1}^L N_\ell^{\frac{\gamma-\beta}{2}}$   
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Homework: complete this proof.



## Examples

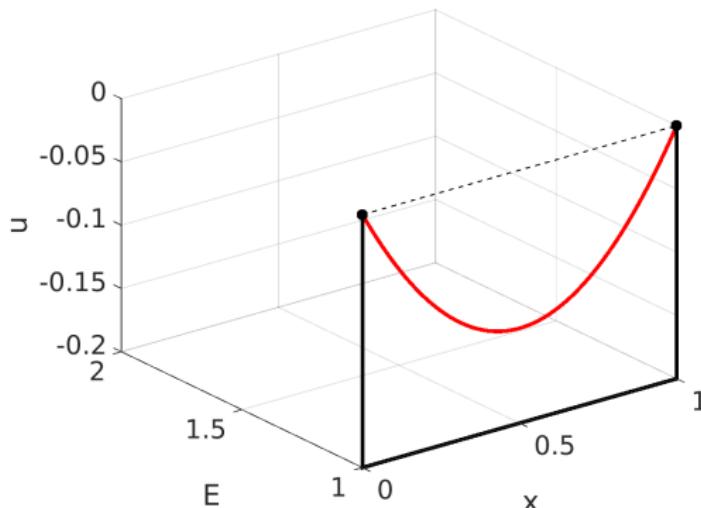
**Model:**

Wire rope (e.g. overhead power line) in equilibrium

$$-u''(x) = f, \quad \text{for } 0 < x < 1 \quad f = \text{gravitation force (const.)}$$

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$$u(1) = 0.$$



Exact solution:

$$u(x) = \frac{f(x - x^2)}{2}$$

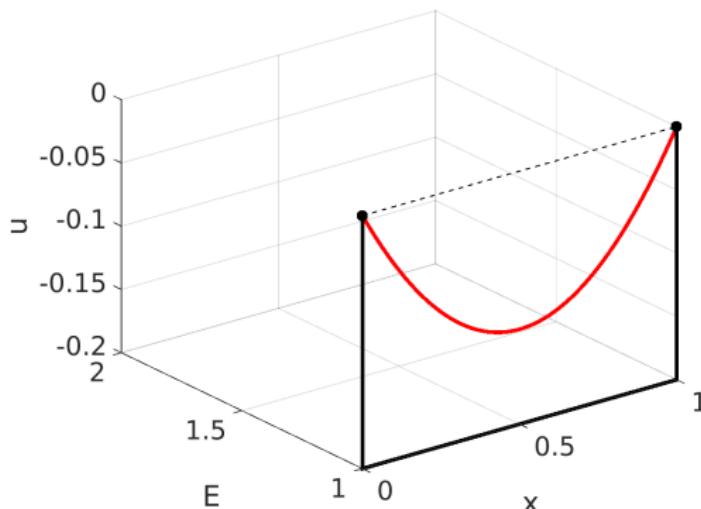
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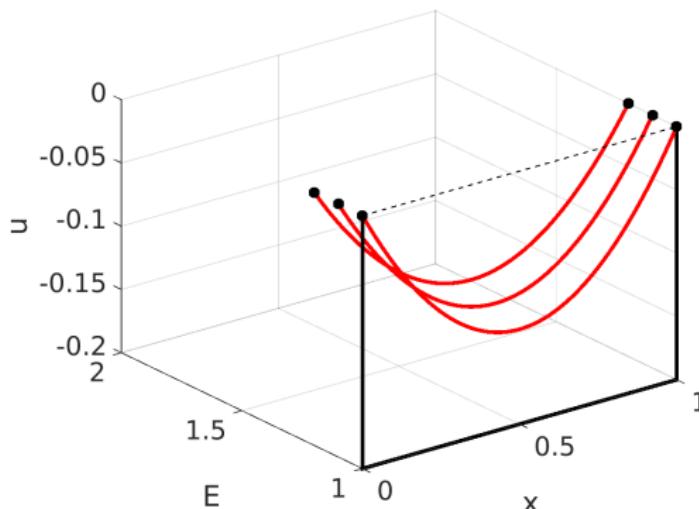
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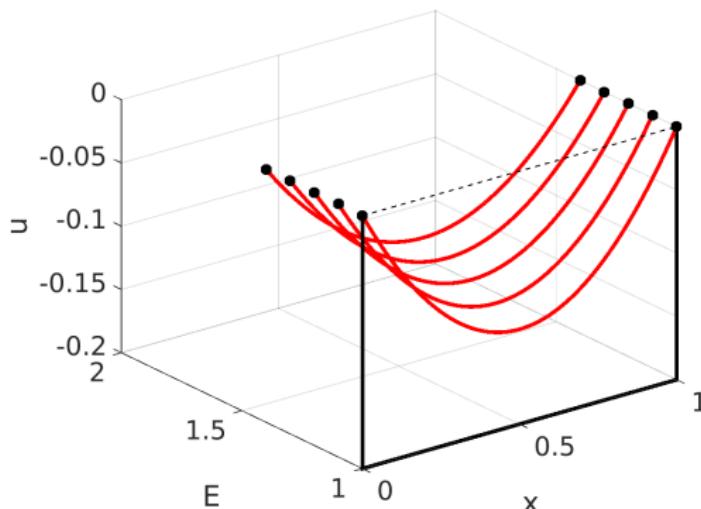
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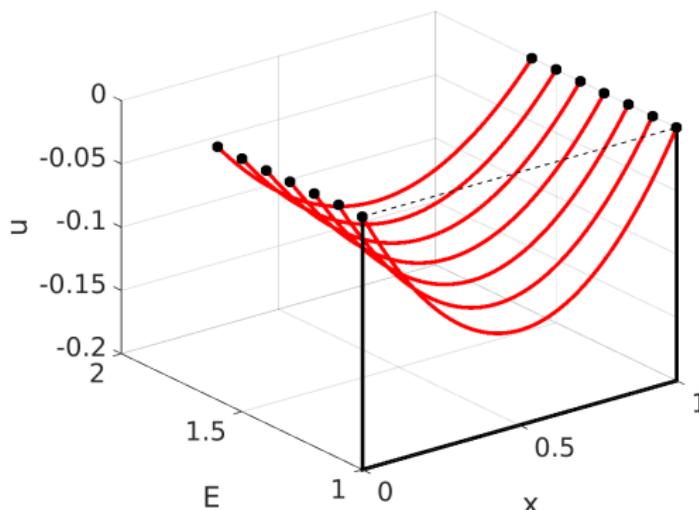
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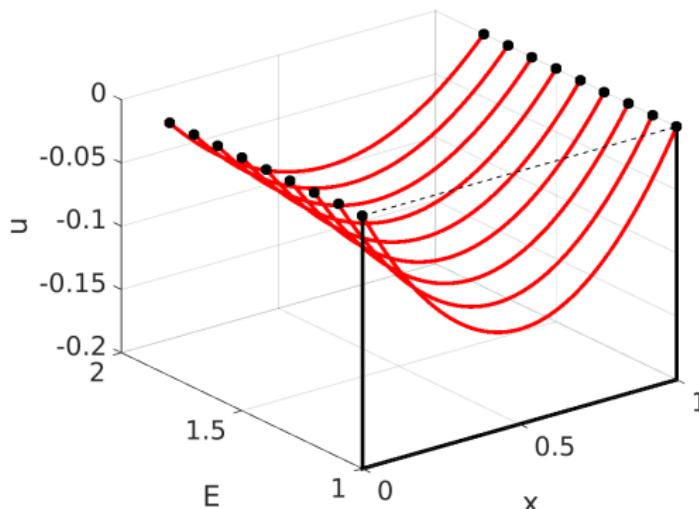
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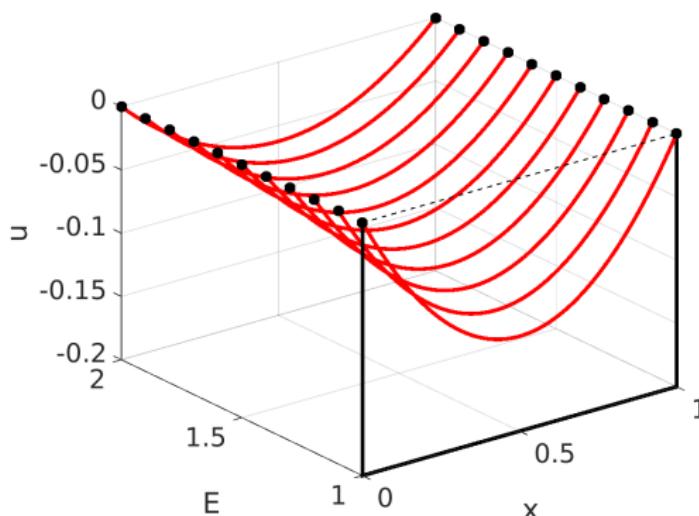
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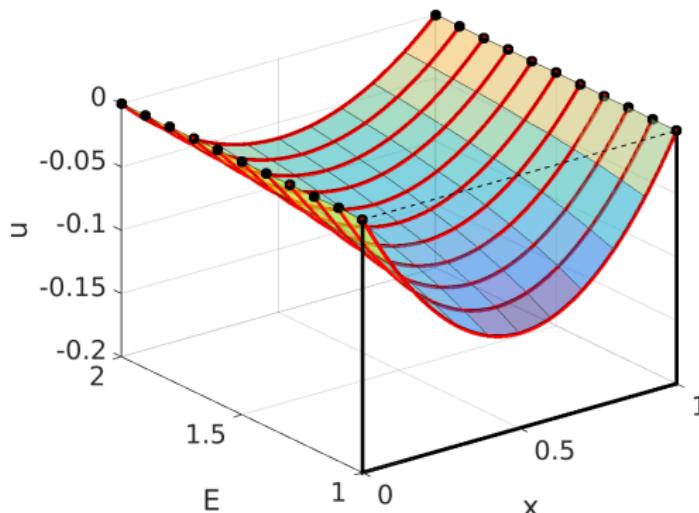
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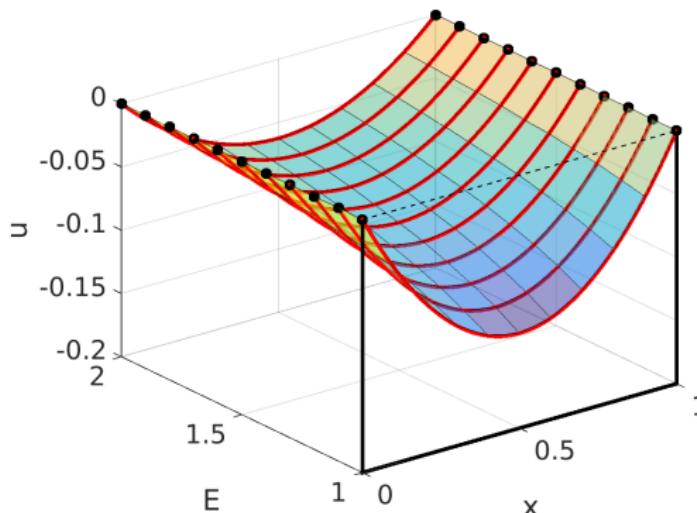
$$u(0) = 0,$$

$$u(1) = 0.$$

$f$  = gravitation force (const.)

$u$  = vertical displacement

$E$  = Young's Modulus



Exact solution:

$$u(x, E) = \frac{f(x - x^2)}{2E}$$

When  $E$  is variable,  $u$  can be viewed as a function of  $x$  and  $E$ .

The variation of  $E$  describes e.g. different materials.

**Example:** Wire rope (conductor) in the electrical overhead line:

- Aluminium
- Steel
- Copper
- Alloys (Aldrey: 99% Al + 0.5% Mg + 0.5% Si)

Variations of the proportion  $\Rightarrow$  Variations of  $E$ .

Typical problem in **forward uncertainty propagation**

Assuming that statistical variations of  $E$  can be estimated in the fabrication process, **is it possible to find probabilistic properties of the wire rope?**

Yes!

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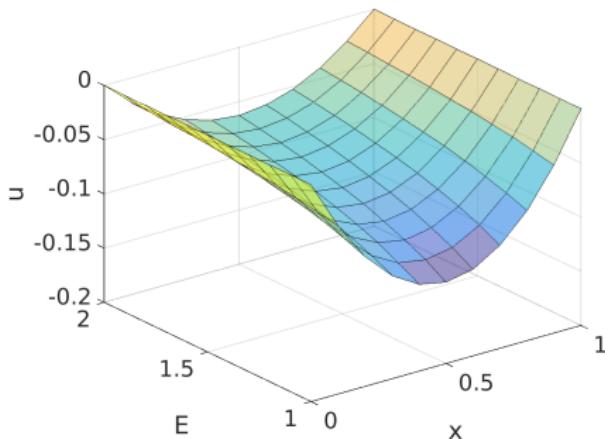
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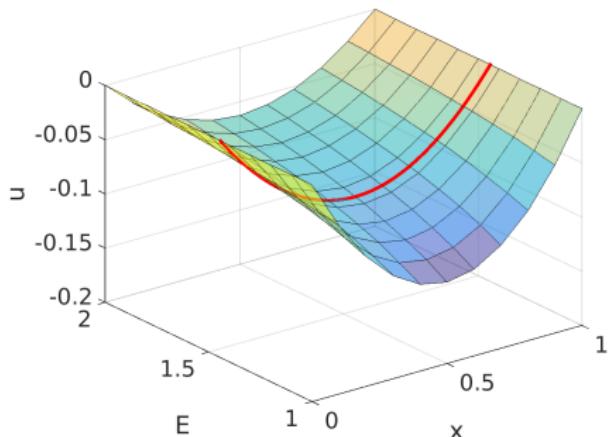


$$u(x, E) = \frac{x^2 - x}{2E}$$

(here  $f = -1$  is assumed)

*Homework:  
check these relations!*

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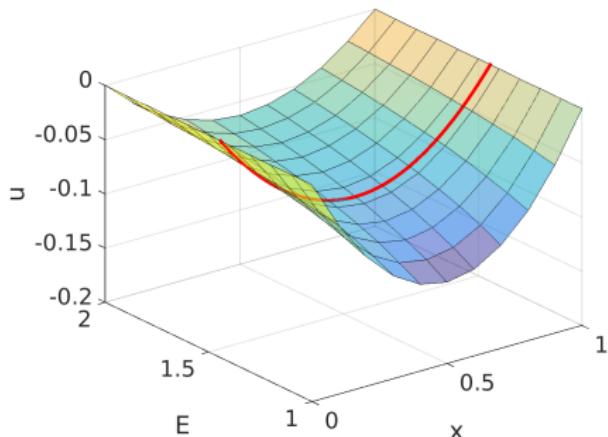
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Mean value:  $\mathbb{E}[u](x) = \frac{x^2 - x}{2} \int_1^2 \frac{dE}{E} = \frac{x^2 - x}{2} \ln 2,$

$$\mathbb{E}[u^2](x) = \left( \frac{x^2 - x}{2} \right)^2 \int_1^2 \frac{dE}{E^2} = \left( \frac{x^2 - x}{2} \right)^2 \frac{1}{2},$$

Variance:  $\text{Var}[u](x) = \left( \frac{x^2 - x}{2} \right)^2 \left( \frac{1}{2} - (\ln 2)^2 \right) =: \sigma(x)^2,$

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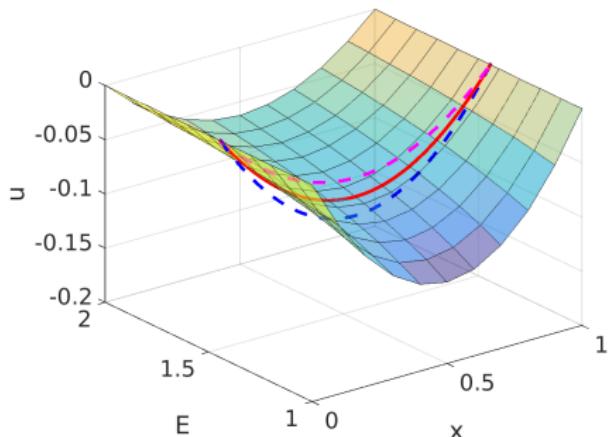
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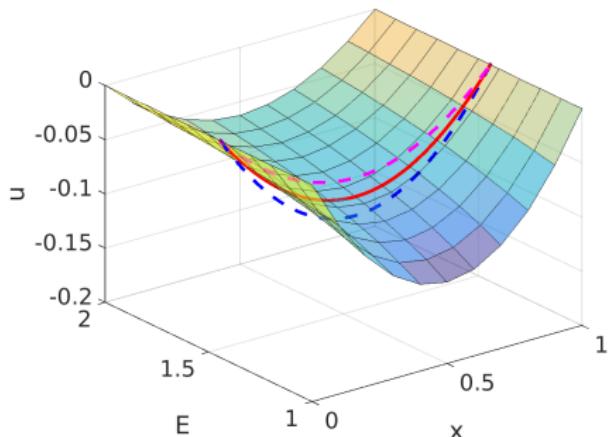
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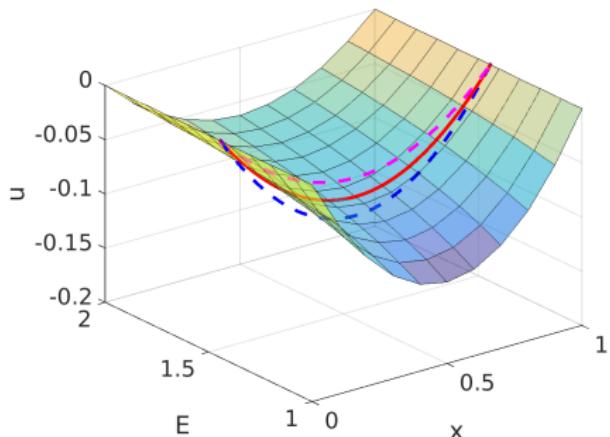
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Autocorrelation:  $\text{Cov}[u](x, y) = \frac{x^2 - x}{2} \frac{y^2 - y}{2} \left( \frac{1}{2} - (\ln 2)^2 \right),$

Correl. Coefficient:  $r(x, y) = \frac{\text{Cov}[u](x, y)}{\sigma(x)\sigma(y)} = 1, \quad (\text{perfect Correlation})$

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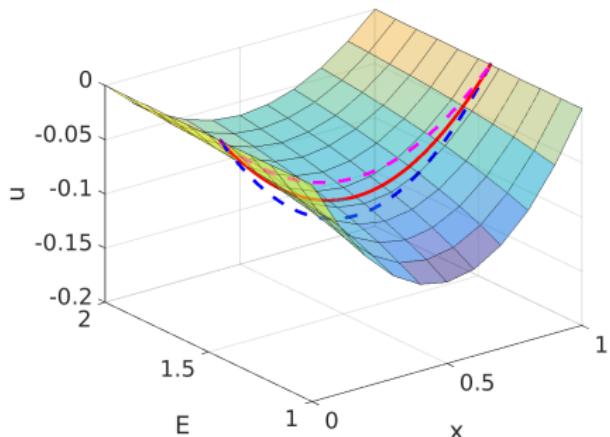
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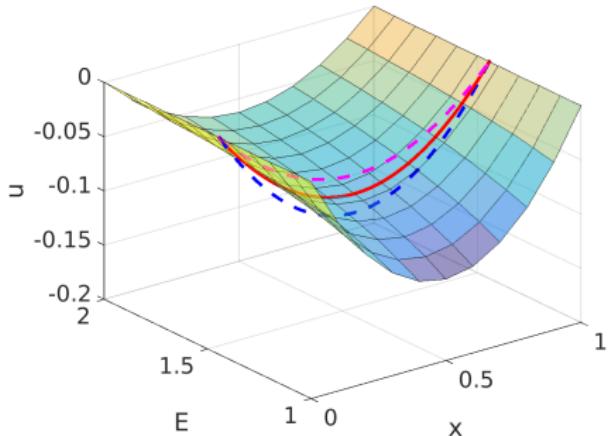
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This is very rare in praxis! The model problem was just too simple:

- The physical domain  $D = (0, 1)$  was one-dimensional;
- $E$  was homogeneous. What if  $E = E(x, \omega)$  varies in space?
- The material law was very simple;
- The solution operator was smooth ...

In practical applications exact evaluation of  $u(x, E)$  is out of reach.

Computer approximations:

$$u(x, E) \approx u_N(x, E) = S_N(E)$$

Is it still possible to **approximately** compute probabilistic properties of the exact solution  $u(x, E)$ ?

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## Examples of uncertain parameters in applications

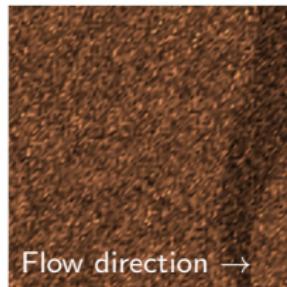
1) Pollution in groundwater flow model:

$$\begin{cases} \mathbf{q} = -K\nabla p & \text{Darcy's law} \\ \nabla \cdot \mathbf{u} = 0 & \text{Mass conservation} \\ \mathbf{q} = \phi\mathbf{u} \end{cases}$$

$\mathbf{q}$  : Darcy flux,  $K$  : conductivity,  $p$  : pressure  
 $\mathbf{u}$  : pore velocity,  $\phi$  : porosity,  $\mathbf{x}$  : position

Particle transport

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x}) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$



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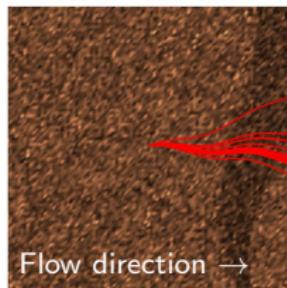
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Random conductivity  
 $K = K(\mathbf{x}, \omega)$

Qty of interest:  
 $T(\omega) = \max\{t : \mathbf{x}(\omega) \in \mathbf{D}\}$   
(particle travel time),  
 $\mathbb{E}[T], \mathbb{V}[T]$

## Examples of uncertain parameters in applications

### 2) Elastic deformation of random media

$$\left\{ \begin{array}{l} \operatorname{div}\sigma + \vec{f} = 0 \\ \sigma_{ij} = \frac{E}{1+\nu} \left( \frac{\nu \delta_{ij} \varepsilon_{kk}}{1-2\nu} + \varepsilon_{ij} \right) \\ \varepsilon = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) \end{array} \right. \begin{array}{l} \text{Equilibrium eq.} \\ \text{Constitutive eq.} \\ \end{array}$$

$\sigma$  : stress,       $\varepsilon$  : strain,       $\mathbf{u}$  : displacement,  
 $\vec{f}$  : volume forces,     $E$  : Young's Modulus,     $\nu$  : Poisson's ratio

Random material parameters:

$$E = E(\mathbf{x}, \omega), \quad \nu = \nu(\mathbf{x}, \omega),$$

Qty of interest:

$$\sigma_{\max}(\omega) = \max_{\mathbf{x} \in D} \{ \|\operatorname{dev}\sigma\|_F \}$$

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---

$$3) + \text{elasto-plastic deformations: } f_{pl} = \|\operatorname{dev}\sigma\|_F - \sqrt{\frac{2}{3}}\sigma_Y \leq 0$$

Random yield stress:

$$\sigma_Y = \sigma_Y(\mathbf{x}, \omega)$$

Qty of interest:

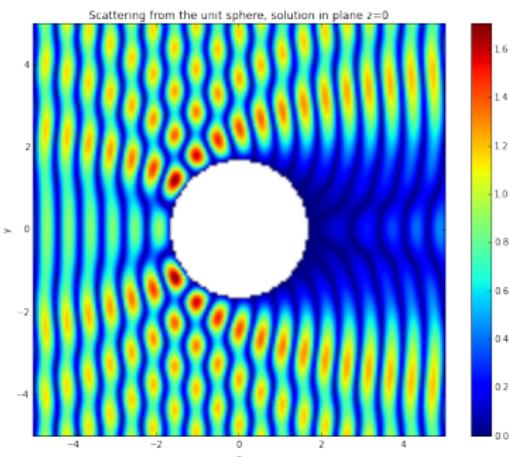
$$\text{Vol}\{f_{pl} = 0\}.$$

## Examples of uncertain parameters in applications

4) Acoustic scattering of objects having uncertain shape

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \setminus D \\ \frac{\partial u}{\partial n} - ik u = g & \text{on } \Gamma := \partial D \end{cases}$$

$u$  : pressure,  $k$  : wave number



Uncertain shape:

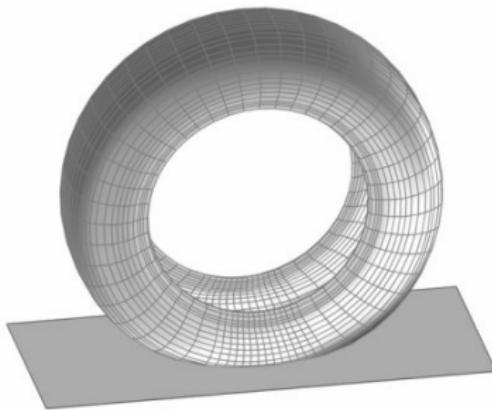
$$\Gamma = \Gamma(\omega)$$

Qty of interest:

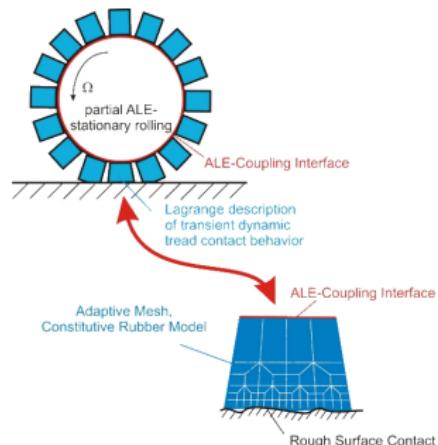
$$U_0(\omega) = u(\mathbf{x}, \omega).$$

(Source: BEM++, T. Betcke et al., [www.bempp.org](http://www.bempp.org))

## 5) Rolling tire on the road: Contact with rough surfaces



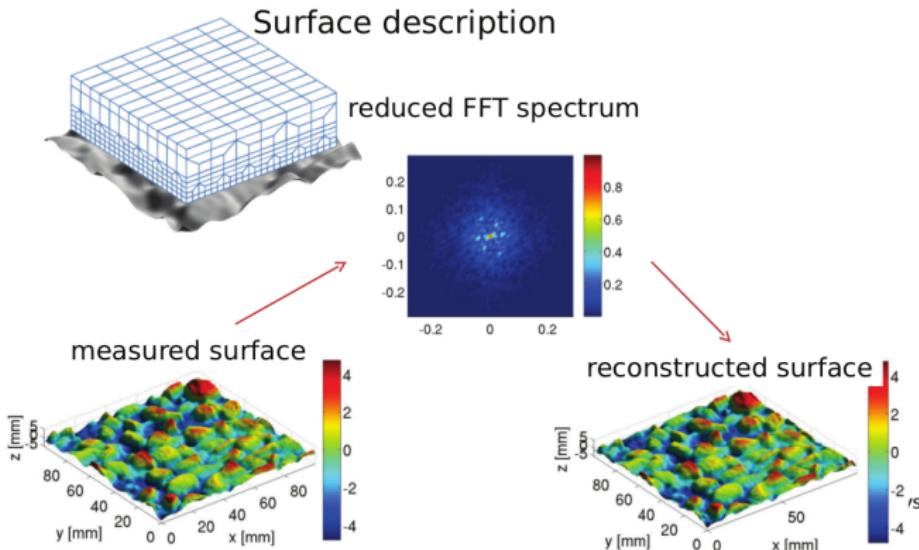
$$\psi(x)$$



Courtesy: Prof. Udo Nackenhorst, IBNM, Univ. Hannover

Input parameter:  $\psi(x)$  is the road surface profile.  
(irregular microstructure)

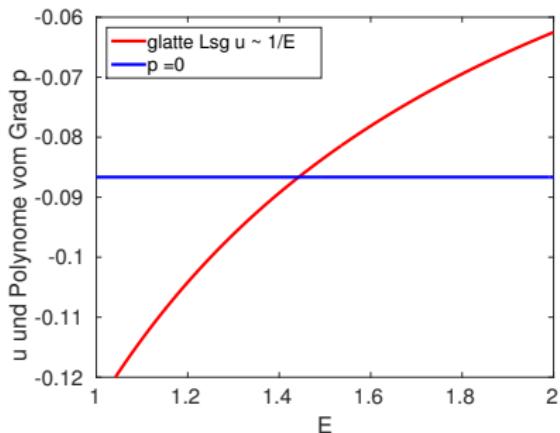
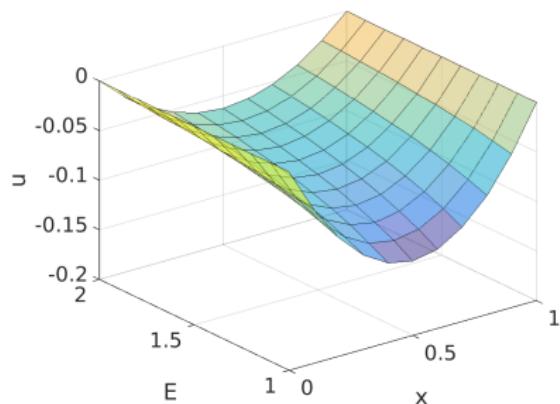
# Rough Surface Contact



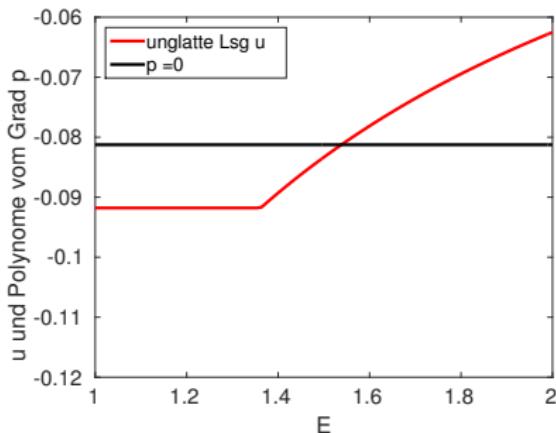
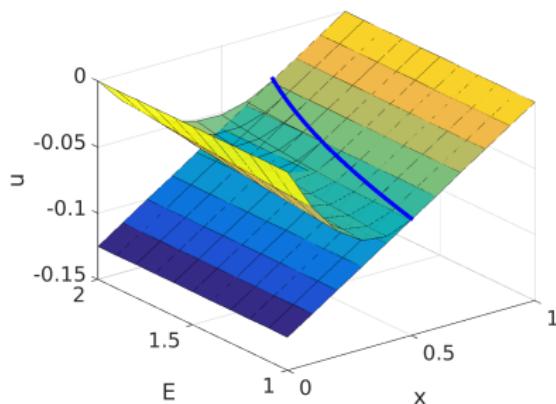
- The road surface  $\psi(x)$  has an irregular microstructure;
- The actual contact zone is a union of a few spots;
- The local microstructure changes as the tire rolls.

# Approximation with Polynomials

Free wire rope

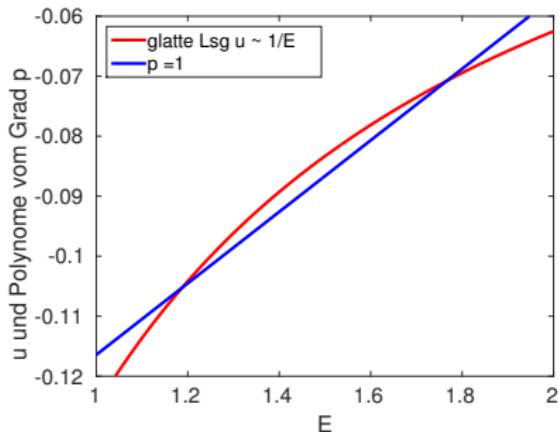
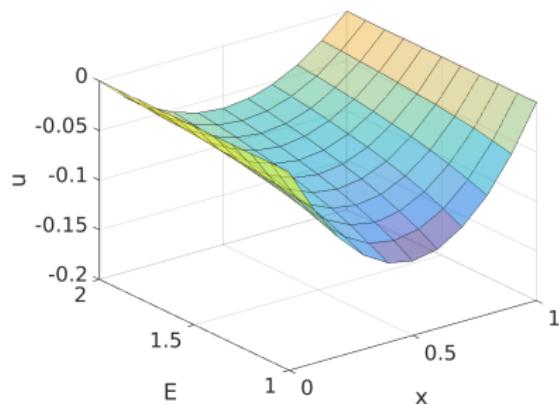


Wire rope with an obstacle

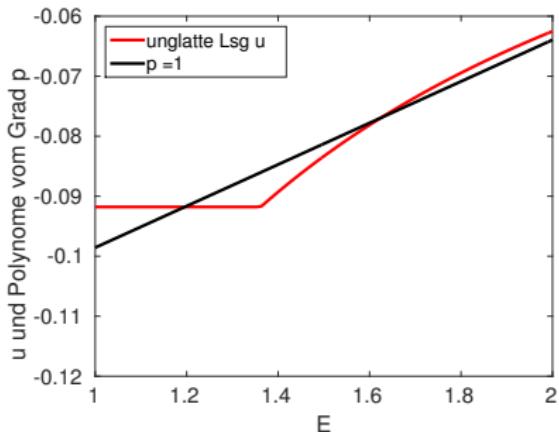
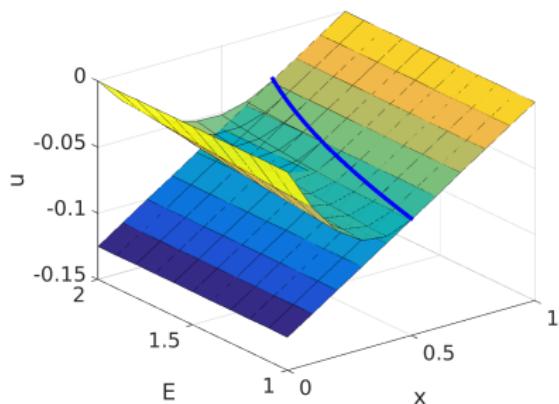


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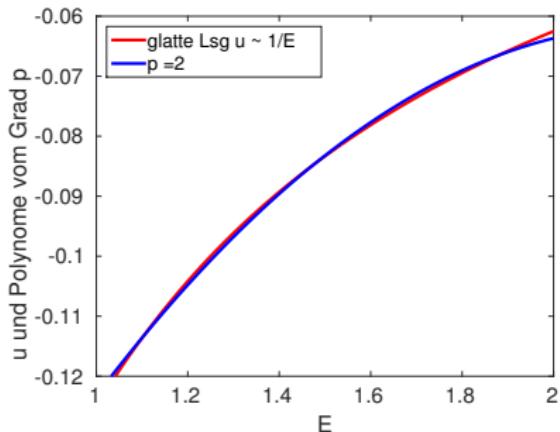
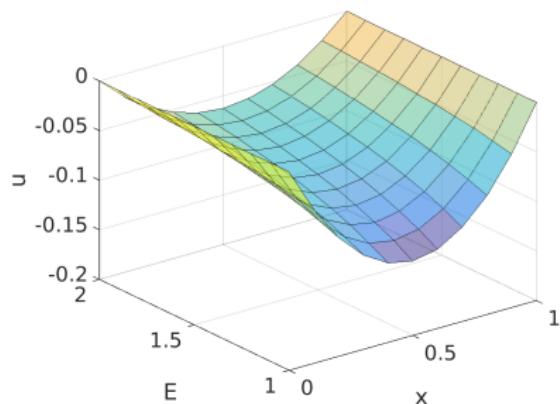


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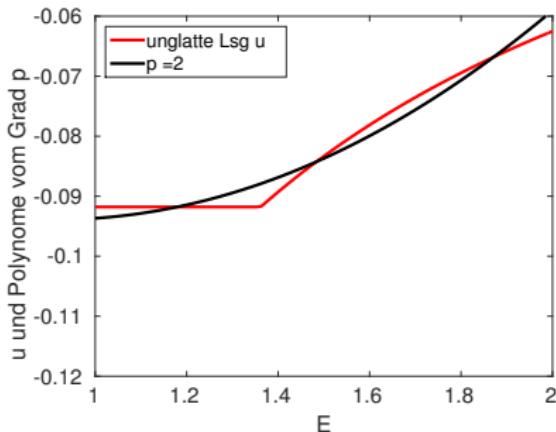
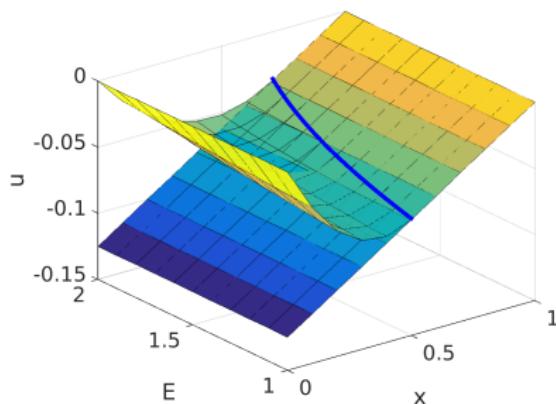


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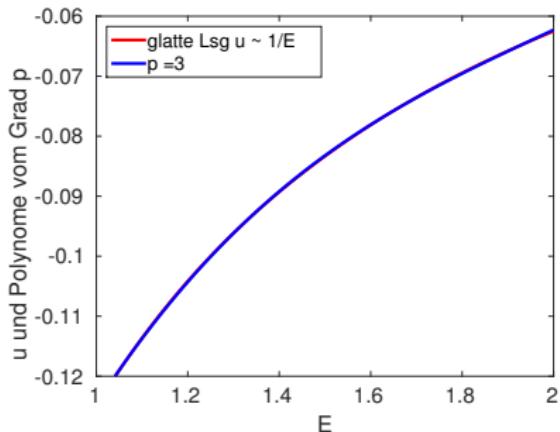
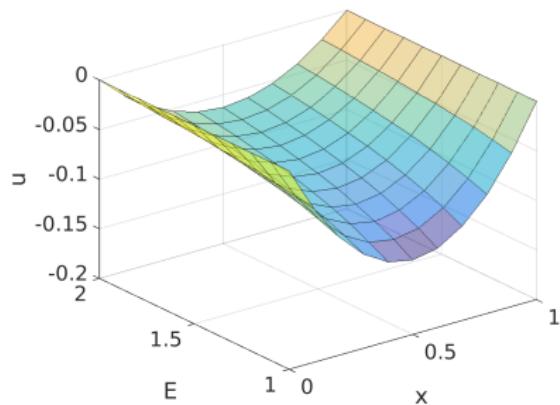


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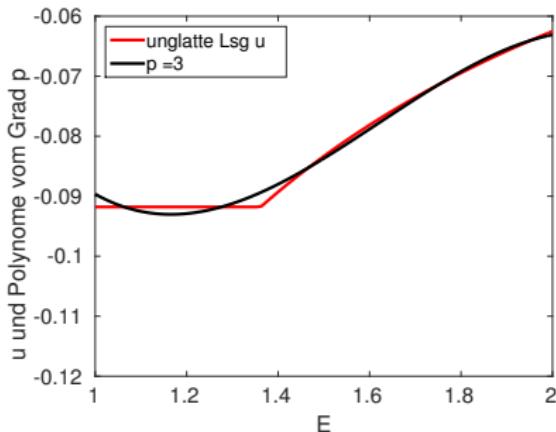
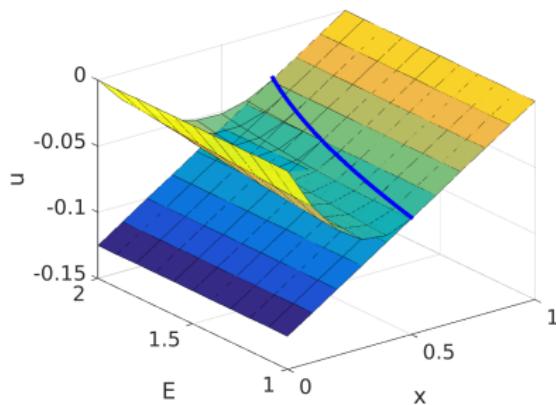


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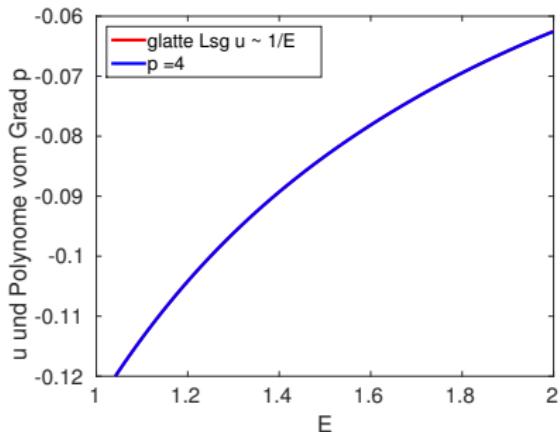
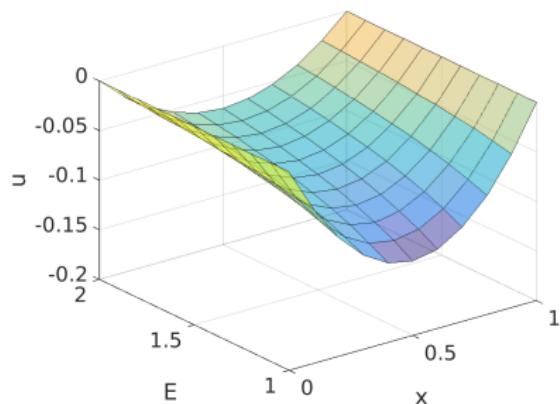


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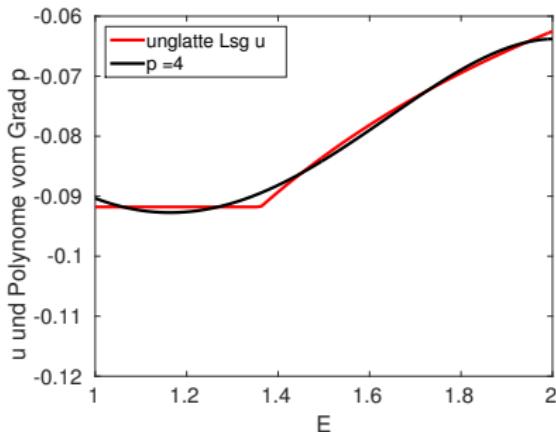
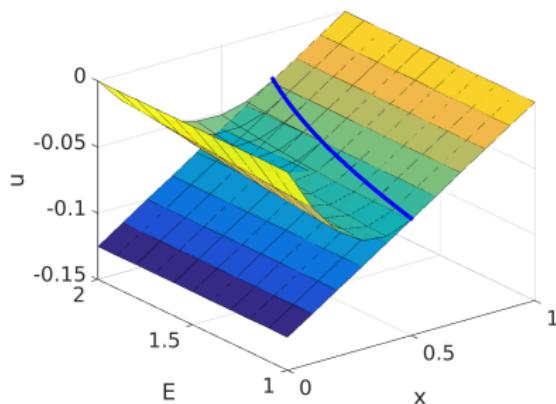


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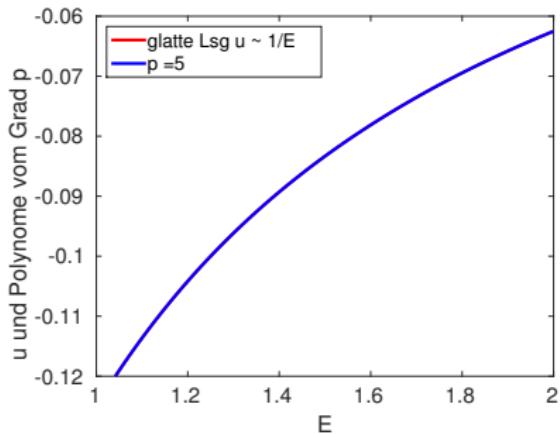
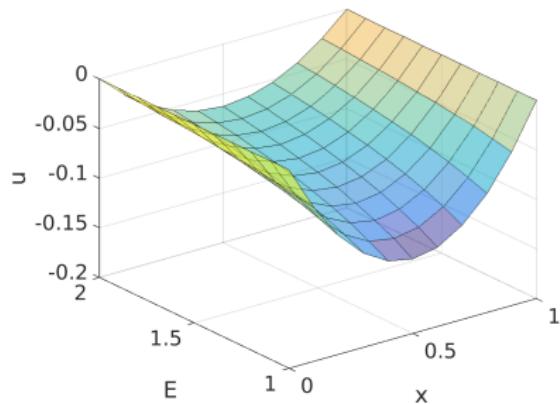


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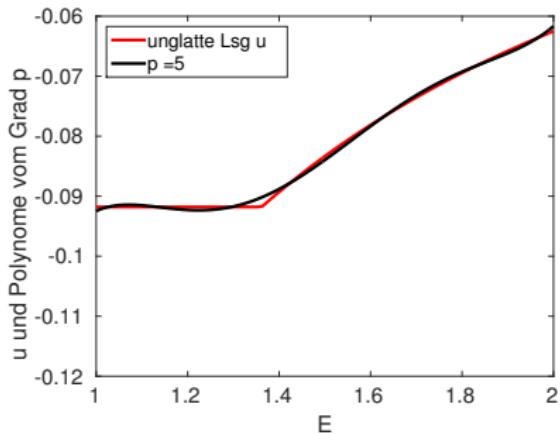
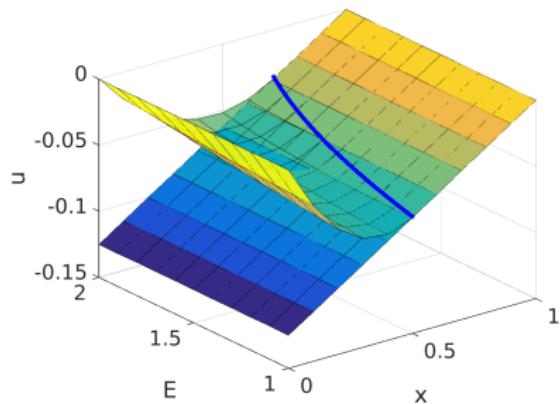


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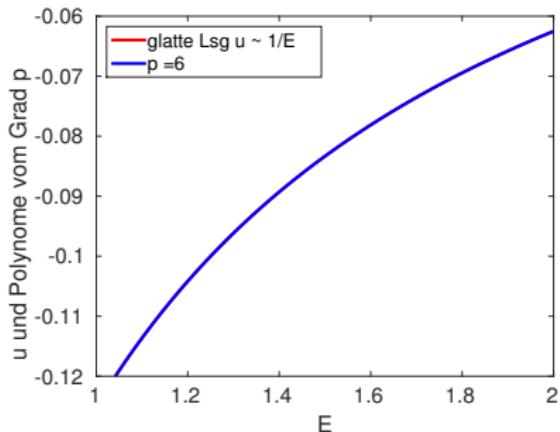
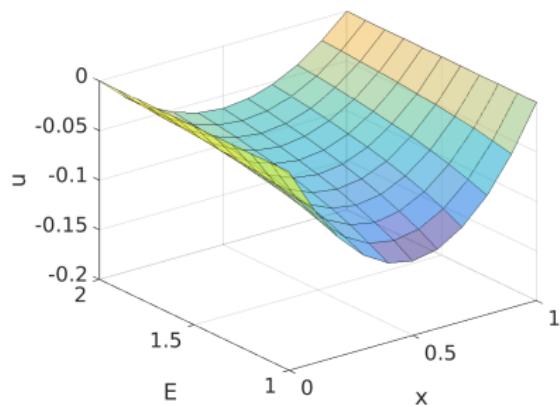


Wire rope with an obstacle

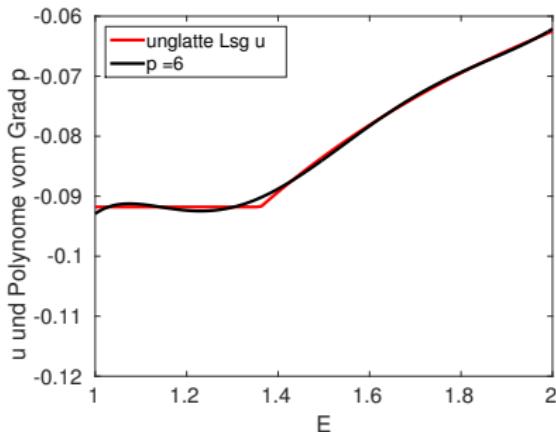
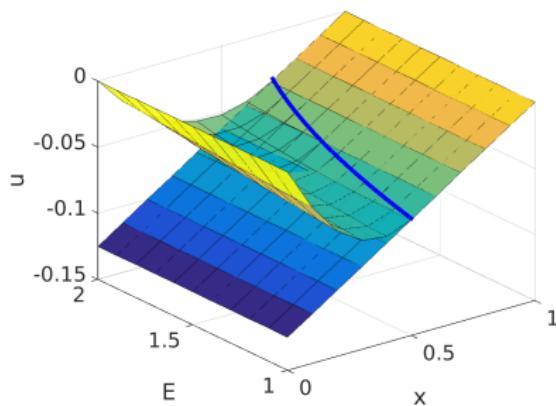


# Approximation with Polynomials

Free wire rope

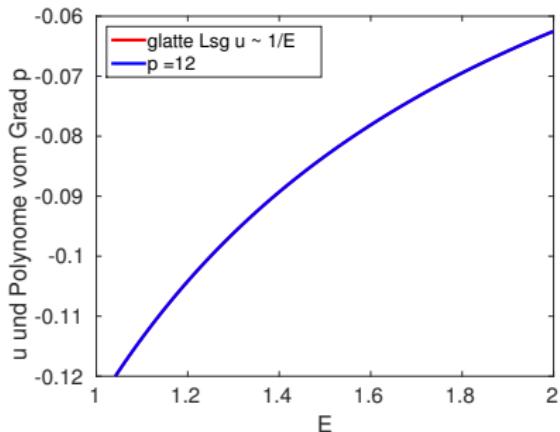
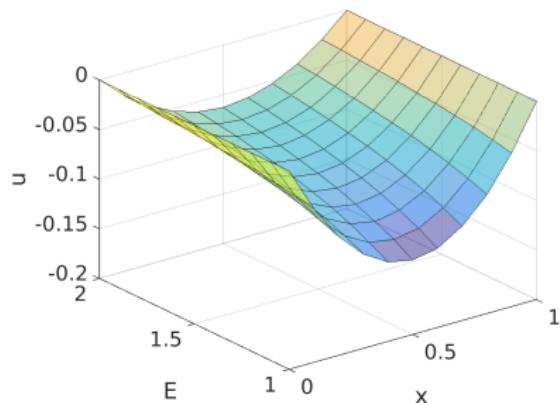


Wire rope with an obstacle

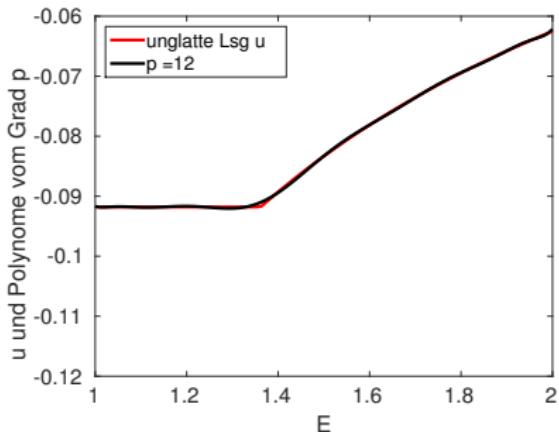
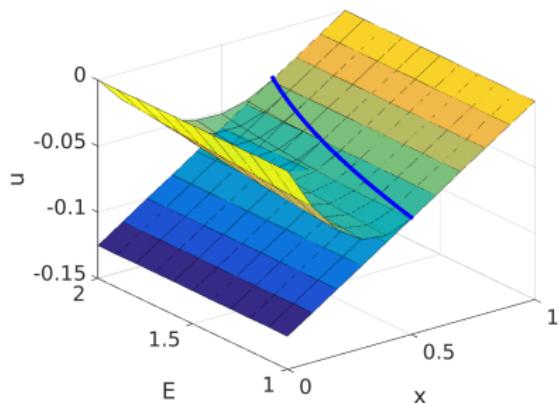


# Approximation with Polynomials

Free wire rope

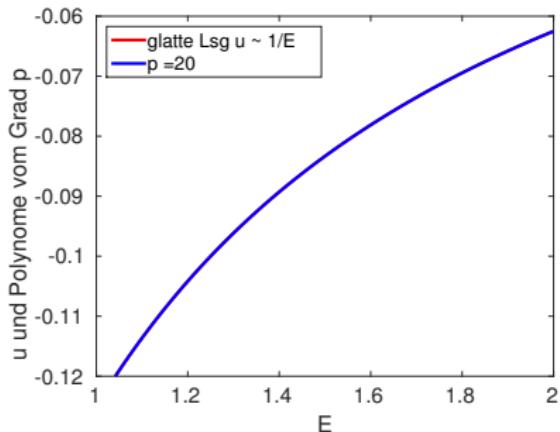
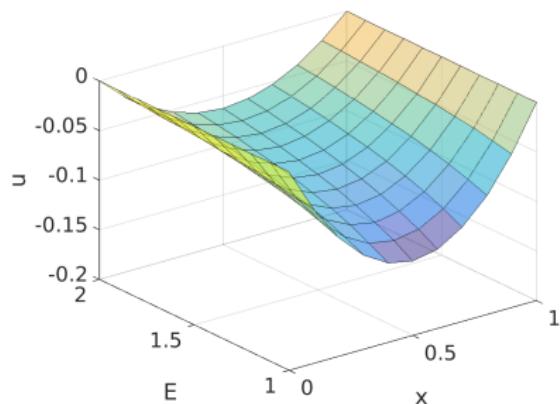


Wire rope with an obstacle

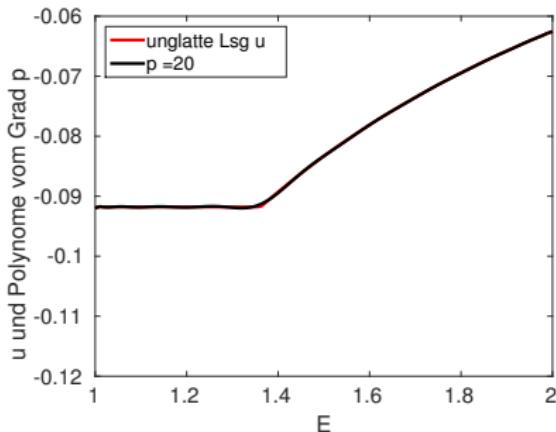
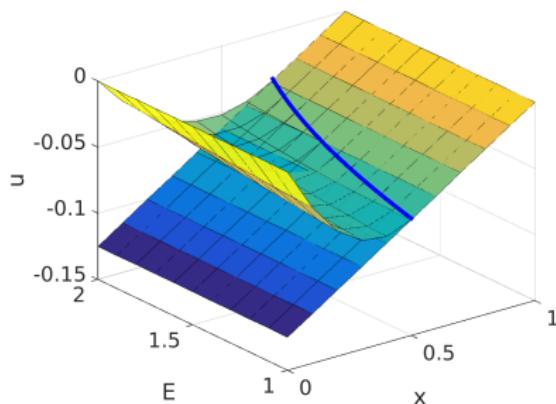


# Approximation with Polynomials

Free wire rope

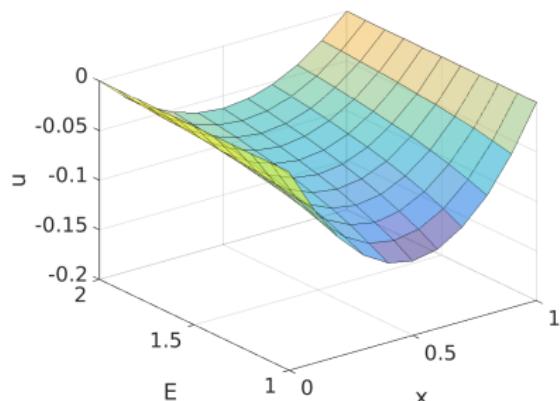


Wire rope with an obstacle

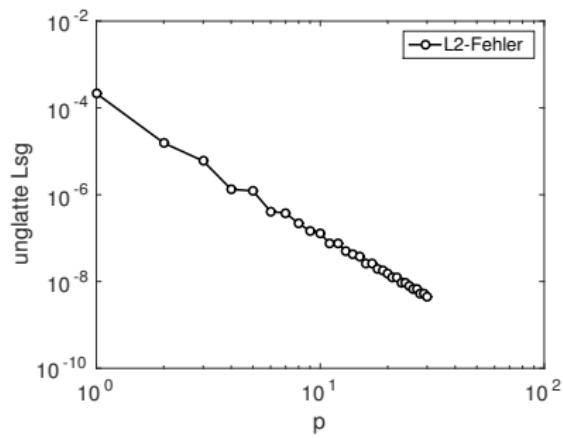
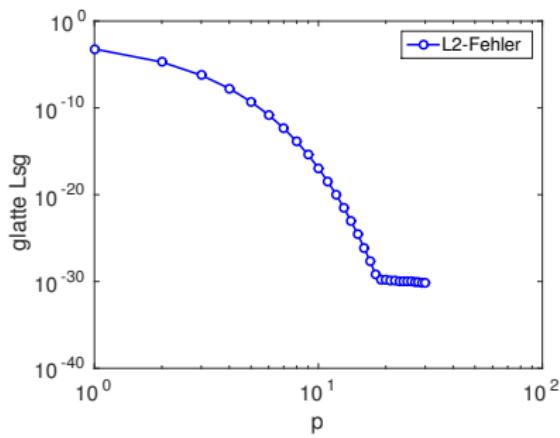
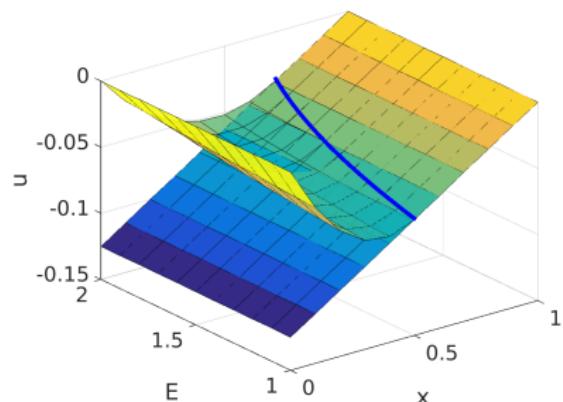


# Approximation with Polynomials

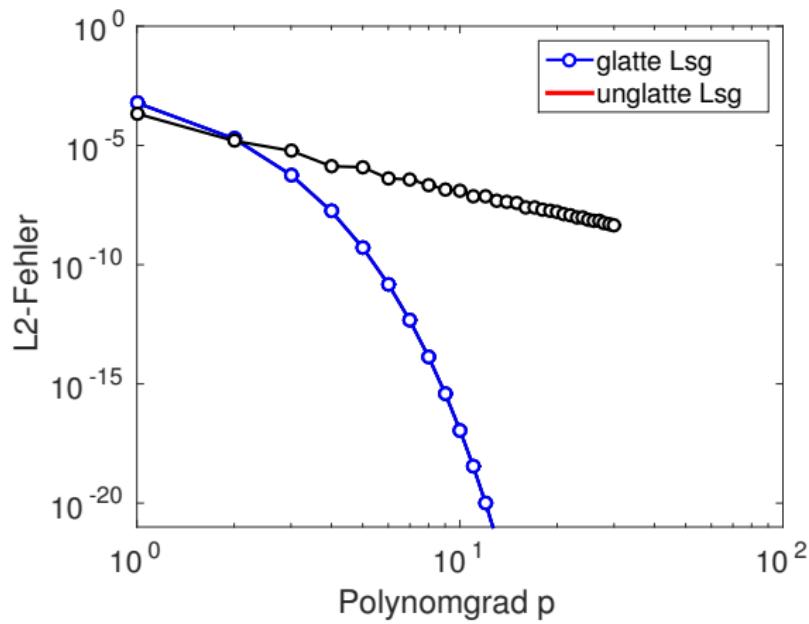
Free wire rope



Wire rope with an obstacle

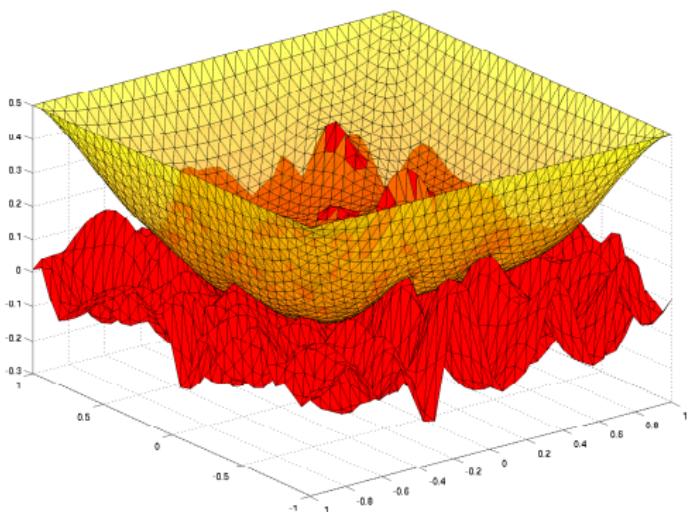


## Approximation with Polynomials



## Example: Contact of an elastic membrane with a rough surface (2d)

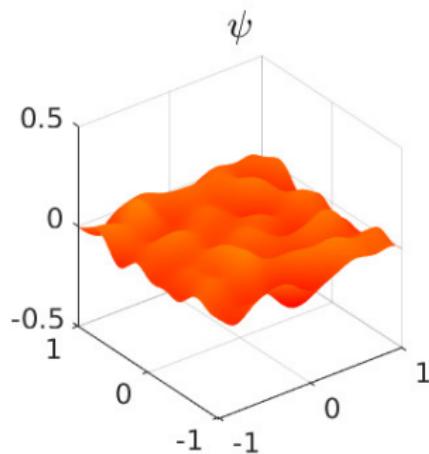
$$\begin{aligned} -\Delta u \geq f, \quad u \geq \psi, \\ (\Delta u + f)(u - \psi) = 0, \end{aligned} \quad \left. \begin{array}{l} \text{in } D, \\ u = 0 \quad \text{on } \partial D. \end{array} \right.$$



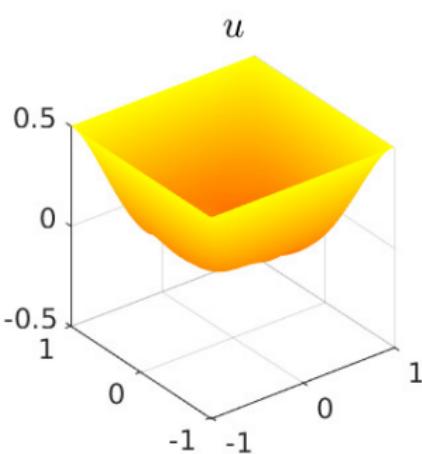
$$\left( \begin{array}{l} \text{here} \\ D = [-1, 1]^2 \end{array} \right)$$

QoI: Deformation  $u(x, \omega)$ ; Contact Area  $\Lambda(\omega) = \{x : u(x, \omega) = \psi(x, \omega)\}$ .

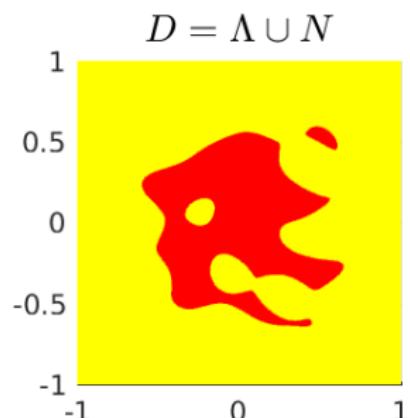
**One realization of the obstacle surface  $\psi = \psi(x)$ :**



Obstacle surface

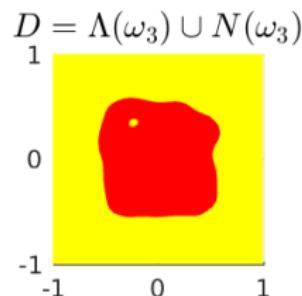
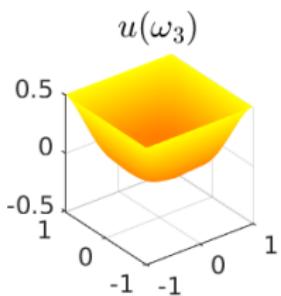
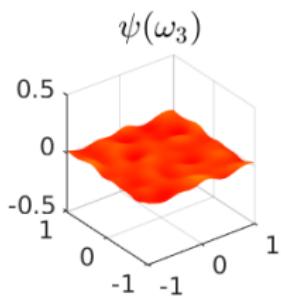
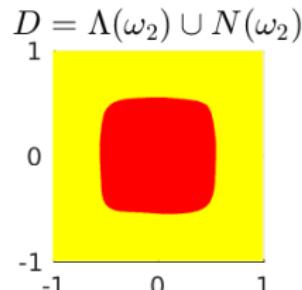
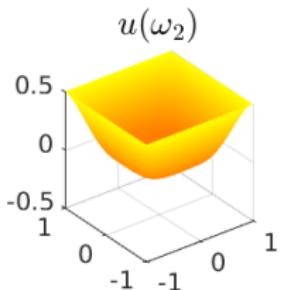
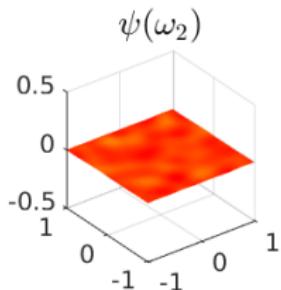
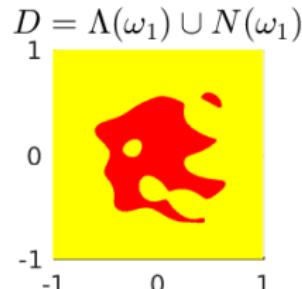
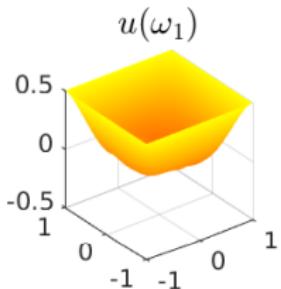
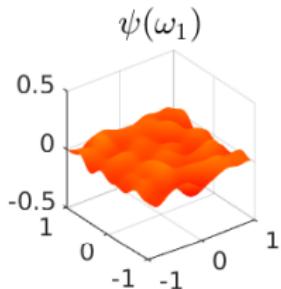


Deformation



Contact set

# Obstacle surfaces of variable/random roughness $\psi = \psi(x, \omega)$ :



## Example: Rough obstacle models

Power spectrum [Persson et al.'05]:

$$\psi(x) = \sum_{q_0 \leq |q| \leq q_s} B_q(H) \cos(q \cdot x + \varphi_q)$$

where  $B_q(H) = \frac{\pi}{5} (2\pi \max(|q|, q_l))^{-H-1} \rightarrow$

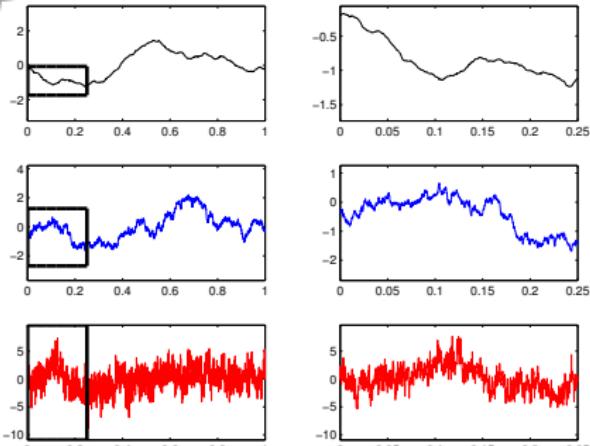
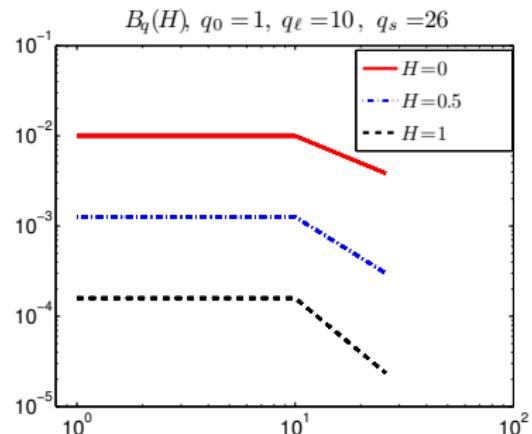
Many materials in Nature and techniques  
obey this law for amplitudes.

---

$H \sim \mathcal{U}(0, 1)$  random roughness

$\varphi_q \sim \mathcal{U}(0, 2\pi)$  random phase

---

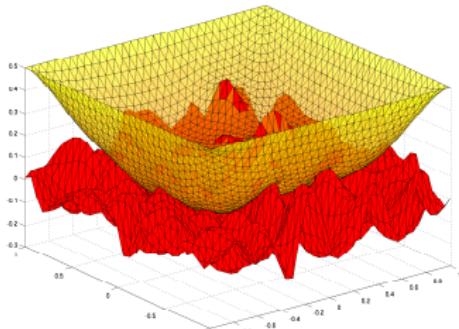


**Forward solver:**

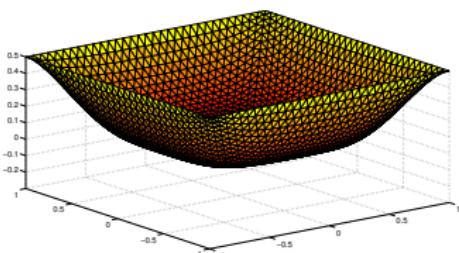
Own implementation of MMG (TNNM)

[Kornhuber'94, ...]

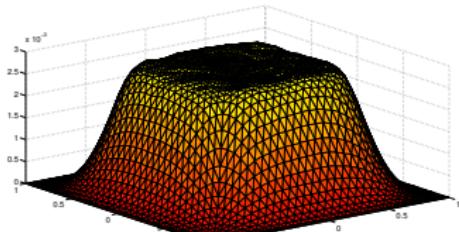
## Approximation of $\mathbb{E}[u]$ and $\text{Var}[u]$ of the deform. field $u(x, \omega)$



A realization of the obstacle  $\psi^i(x)$  and the deformation profile  $u_h^i(x)$



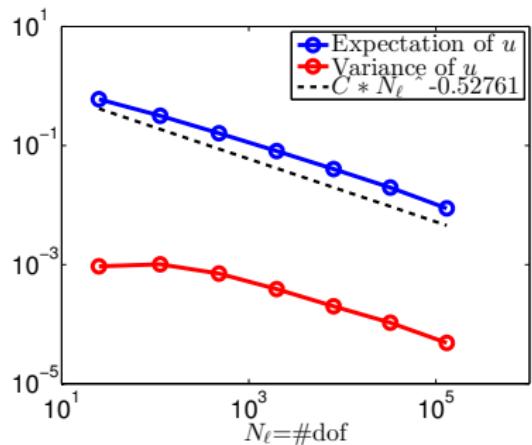
The mean deformation profile  
 $E^{ML}[u]$



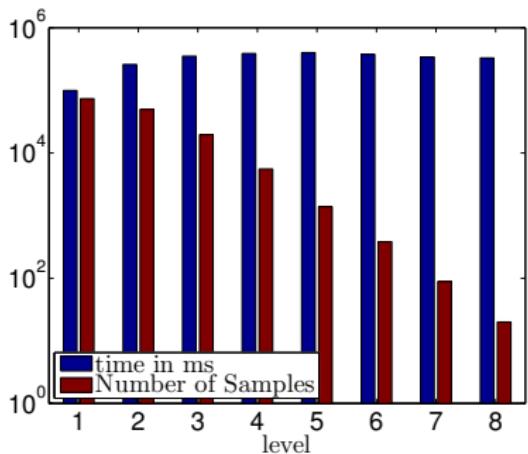
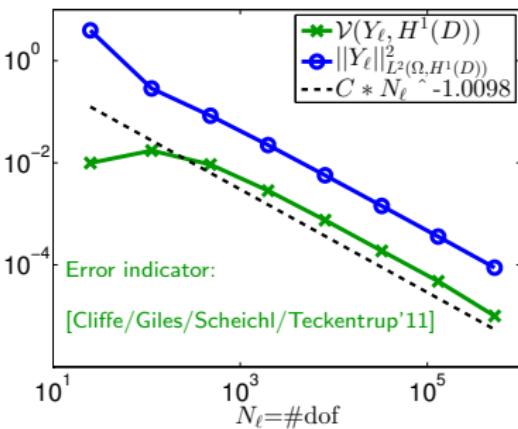
The variance of the deformation  
profile  $V^{ML}[u]$

# Approximation of $\mathbb{E}[u]$ and $\text{Var}[u]$ of the deform. field $u(x, \omega)$

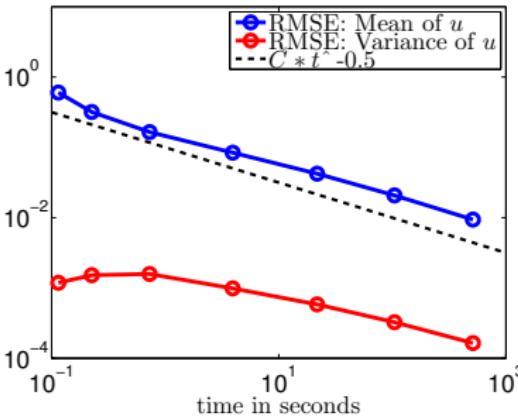
bias of the estimator



variance of the estimator

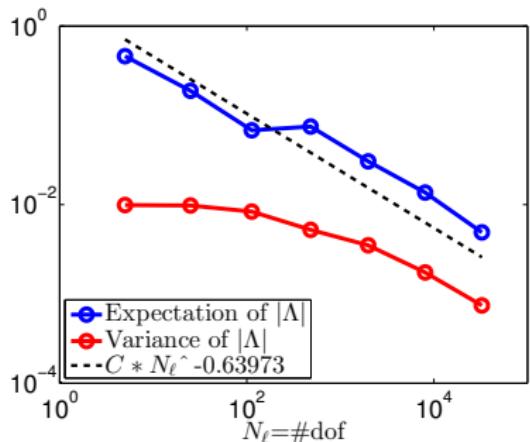


total error vs. runtime

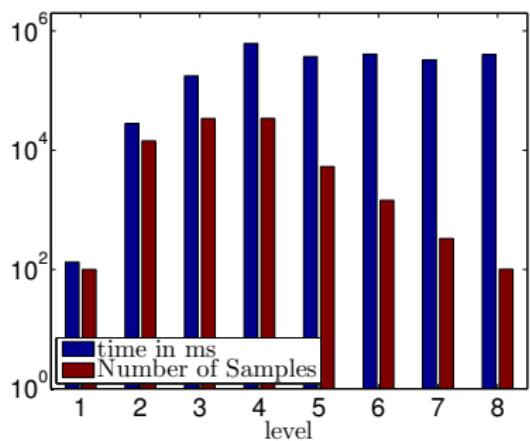
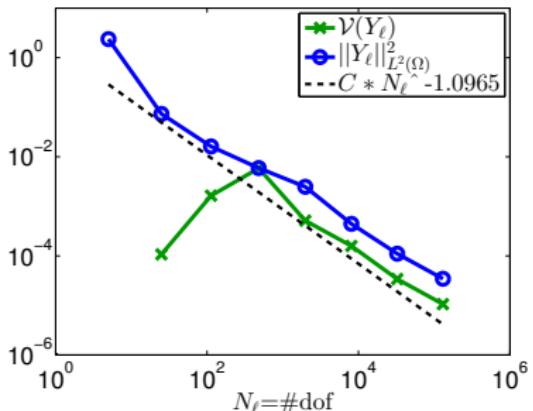


# Approximation of $\mathbb{E}[X]$ and $\text{Var}[X]$ of the contact area $X = |\Lambda|$

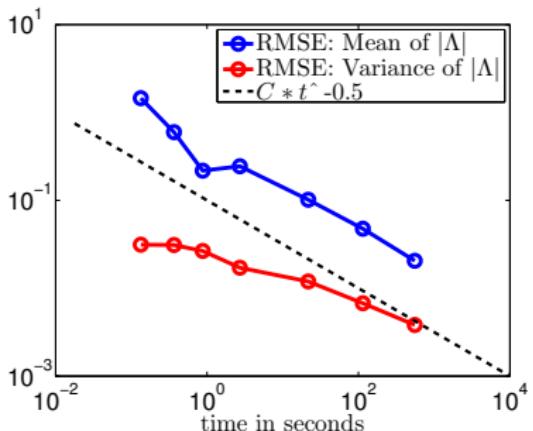
bias of the estimator



variance of the estimator



total error vs. runtime



## Estimators for the Variance:

Recall the mean estimator

$$E^{ML}[X] := \sum_{\ell=1}^L E_{M_\ell}[X_\ell - X_{\ell-1}]$$

where  $E_M[X_\ell] := \frac{1}{M} \sum_{i=1}^M X_\ell^i.$

### Benefits:

- $V^{ML}[X]$  is unbiased, i.e.  $\mathbb{E}\left[V^{ML}[X] - V[X_L]\right] = 0$
- Fast one pass stable evaluation formulae (single level in [Pebay'08])

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... then define the variance estimator by

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where  $V_M[X_\ell] := \frac{1}{M-1} \sum_{i=1}^M (X_\ell^i - E_M[X_\ell])^2.$

see BIERIG, CHERNOV, Numer. Math. (2015)

## Benefits:

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Theorem (a priori estim.: random obstacle problem, [Bierig/AC'15])

Suppose:  $\psi \in L^\infty(\Omega, W^{2,r})$  for some  $r > 2$

Deterministic fwd solver:  $\|u_\ell - u\|_{H^1} \lesssim h_\ell$ , pw. lin. FE

with the Total Work  $\sim \ell^\nu N_\ell$  ( $N_\ell \sim h_\ell^{-2}$ , i.e. lin. cost).

---

Then: **MLMC** with the optimal choice  $M_\ell := (h_\ell/h_L)^2$  satisfies

$$\left. \begin{aligned} \|E^{ML}[u] - \mathbb{E}[u]\|_{L^2(\Omega, H^1)} \\ \|V^{ML}[u] - \mathbb{V}[u]\|_{L^2(\Omega, H^1)} \end{aligned} \right\} \lesssim h_L \sqrt{|\log h_L|},$$

with the Total Work  $\sim L^{\nu+1} N_L$ .

Almost linear complexity for MLMC + MMG.

(Sampling is asymptotically almost for free!)

Theorem (a priori estim.: random obstacle problem, [Bierig/AC'15])

Suppose:  $\psi \in L^{2q}(\Omega, W^{2,2})$  and  $\frac{1}{p} + \frac{1}{q} = 1$

Deterministic fwd solver:  $\|u_\ell - u\|_{H^1} \lesssim h_\ell$ , pw. lin. FE

with the Total Work  $\sim \ell^\nu N_\ell$  ( $N_\ell \sim h_\ell^{-2}$ , i.e. lin. cost).

---

Then: **MLMC** with the optimal choice  $M_\ell := (h_\ell/h_L)^2$  satisfies

$$\|E^{ML}[u] - \mathbb{E}[u]\|_{L^2(\Omega, H^1)} \lesssim h_L \sqrt{|\log h_L|},$$

$$\|V^{ML}[u] - \mathbb{V}[u]\|_{L^2(\Omega, H^1)} \lesssim h_L^{\frac{1}{p}}, \quad (\text{using inv. ineq.})$$

with the Total Work  $\sim L^{\nu+1} N_L$ .

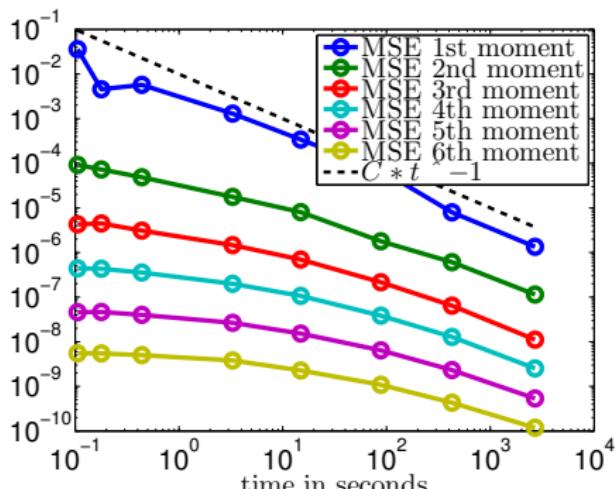
Almost linear complexity for MLMC + MMG.

(Sampling is asymptotically almost for free!)

**Extension to higher order moments:**  $\mathcal{M}^k[X] := \mathbb{E}[(X - \mathbb{E}[X])^k]$

$$S_M^3[X] := \frac{M}{(M-1)(M-2)} \sum_{i=1}^M (X_i - E_M[X])^3 \quad (\text{unbiased})$$

$$S_M^k[X] := \frac{1}{M} \sum_{i=1}^M (X_i - E_M[X])^k \quad (\text{small bias})$$

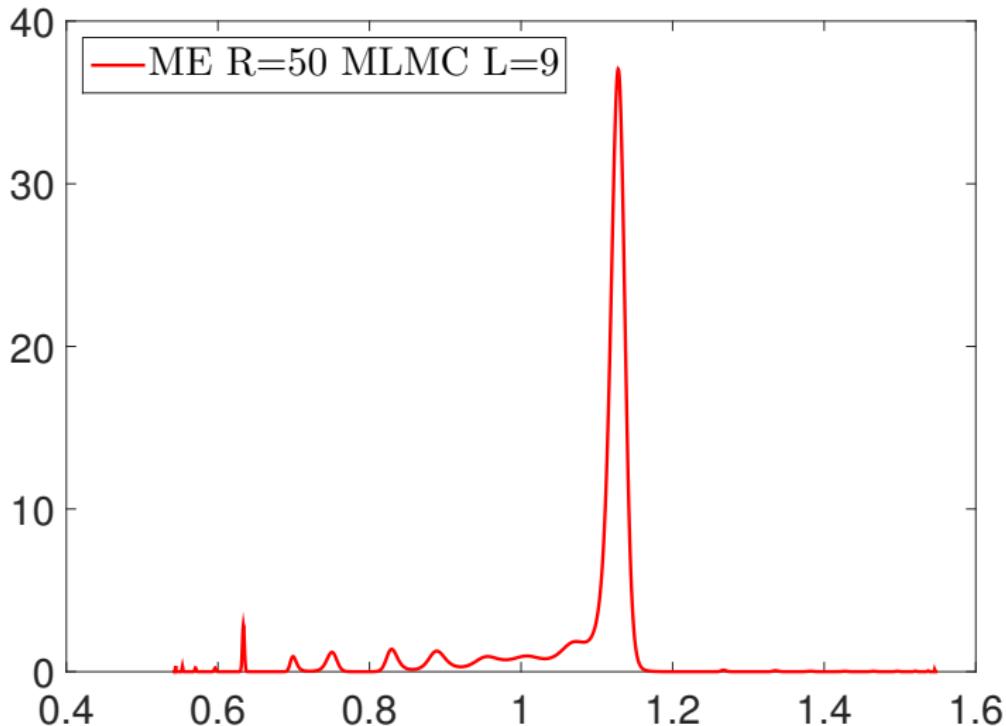


$X = |\Lambda|$ , contact area

Notice:  
 $|\Lambda| \leq |D|$

[BIERIG, CHERNOV,  
JSPDE'16]

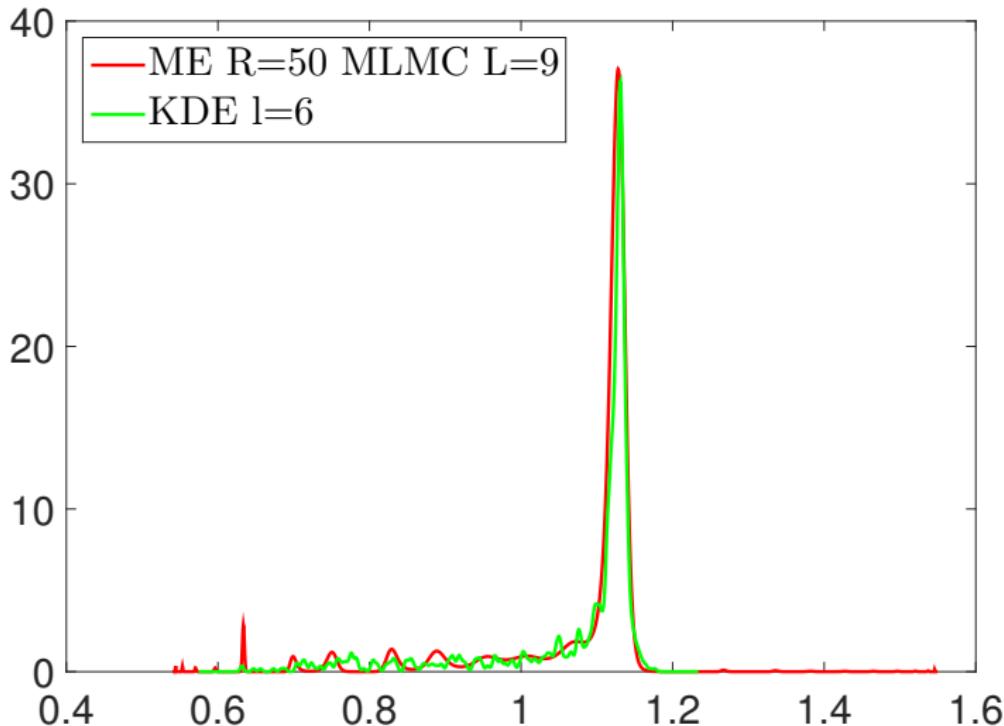
## Estimation of the PDF $\rho_X$ of the contact area $X = |\Lambda|$ by the Maximum Entropy method



The peak(s) corresponds to ca. 28.2% of the membrane in contact with the surface

More experiments and rigorous error analysis in [Bierig/Chernov, JCP'16]

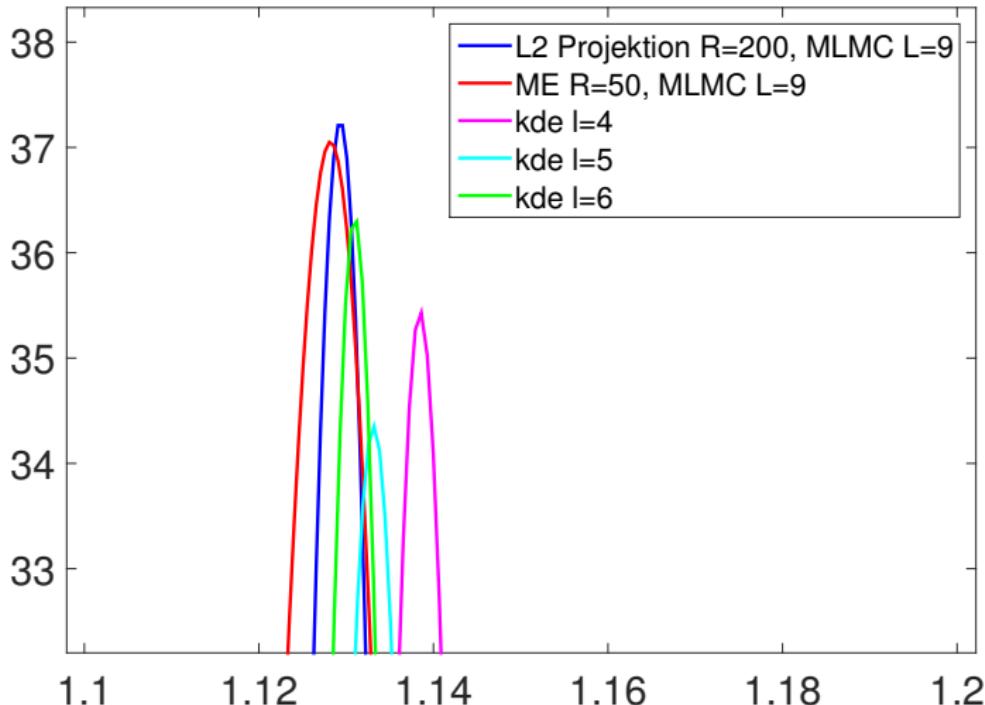
## Estimation of the PDF $\rho_X$ of the contact area $X = |\Lambda|$ by the Maximum Entropy method



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The peak(s) corresponds to ca. 28.2% of the membrane in contact with the surface

More experiments and rigorous error analysis in [Bierig/Chernov, JCP'16]

## Towards adaptivity – adaptive selection of

- the number of moments  $R$
- the interval of approximation  $[a, b]$

### Test example:

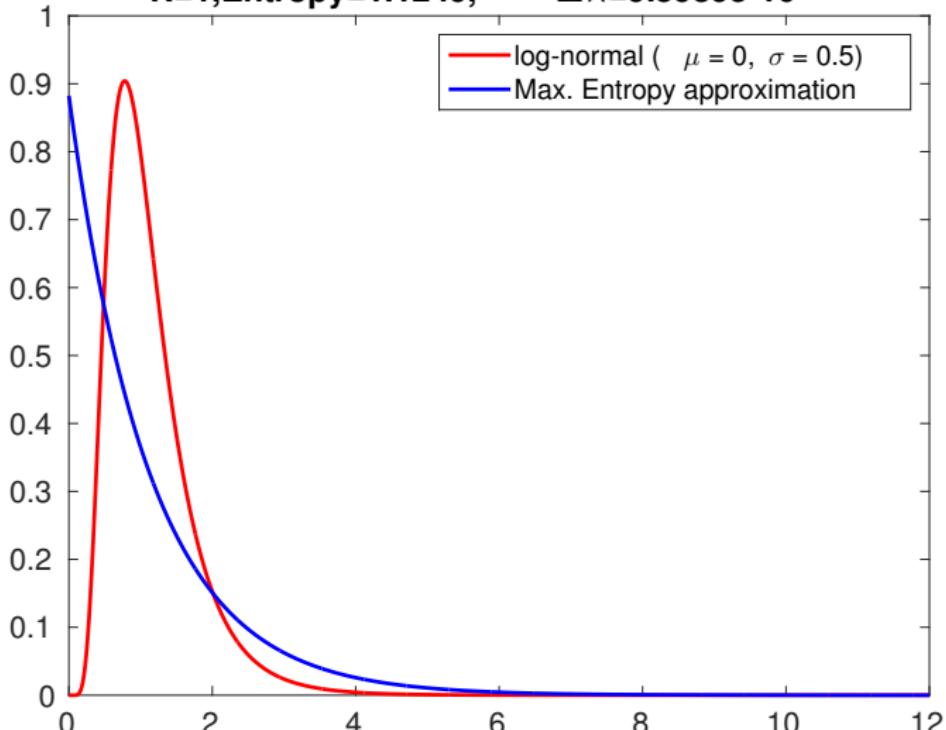
Log-normal distribution with  $\mu = 0$  and variable  $\sigma$  ( $= 0.5$  and  $0.2$ )

Estimation of moments  $\mu_1, \dots, \mu_R$  by MC with  $10^8$  samples

### Stopping parameters for the Newton Method:

- $\Delta\lambda \leq 10^{-9}$  (convergence)
- $\Delta\lambda \geq 10^3$  (no convergence)
- $\#\text{iter} \geq 1000$  (no convergence)

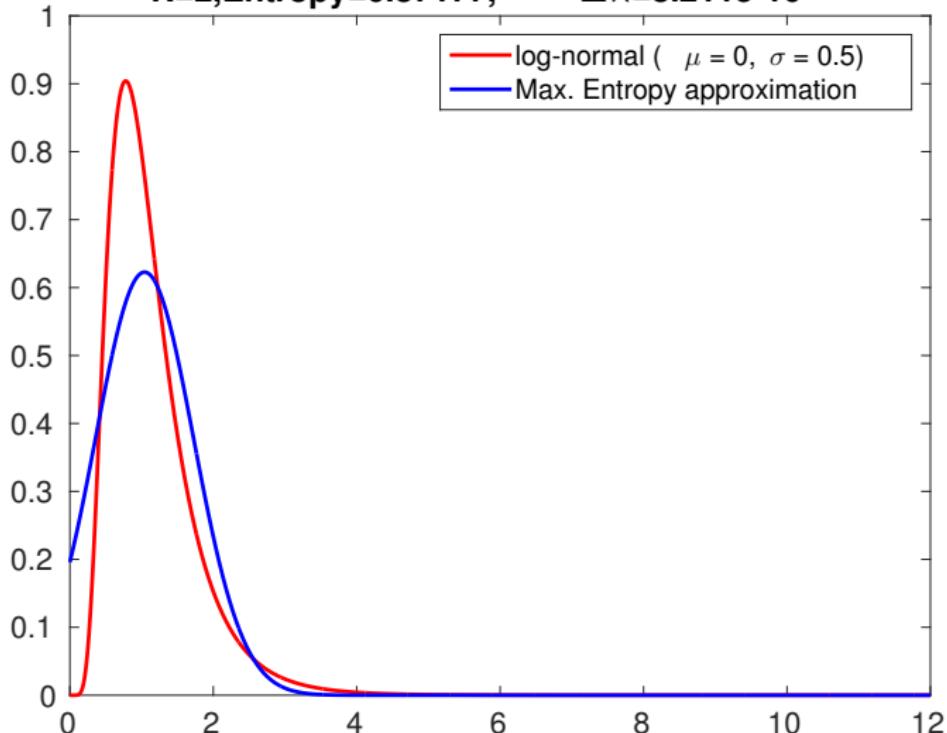
$$R=1, \text{Entropy}=1.1249, \Delta\lambda=9.5939e-10$$



### Legendre Moments:

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

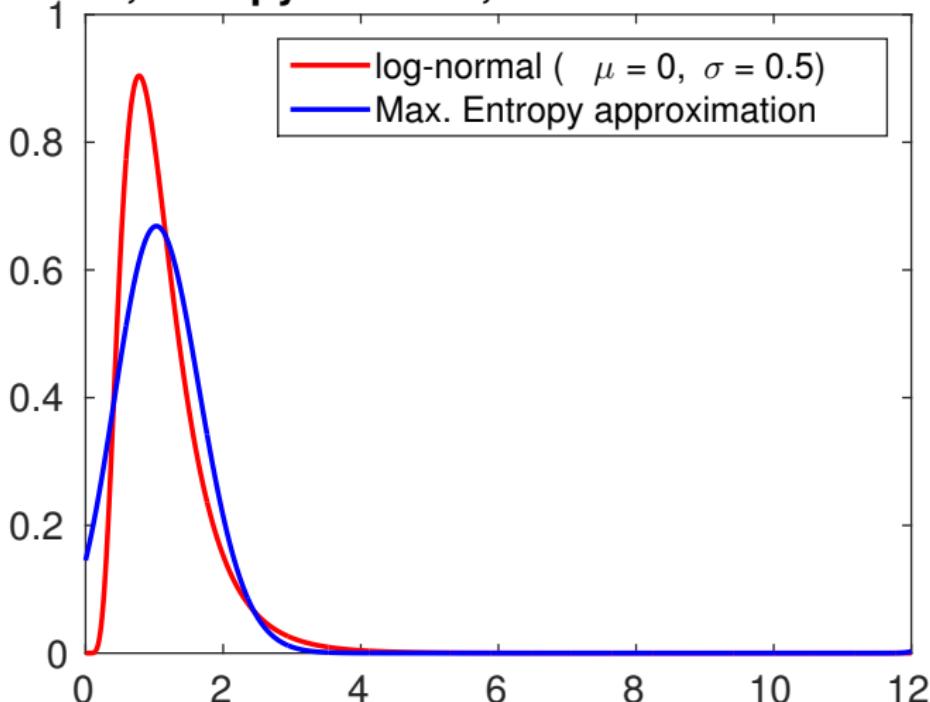
$$R=2, \text{Entropy}=0.87177, \Delta\lambda=8.211e-10$$



### Legendre Moments:

- Stable for  $R \leq 8$
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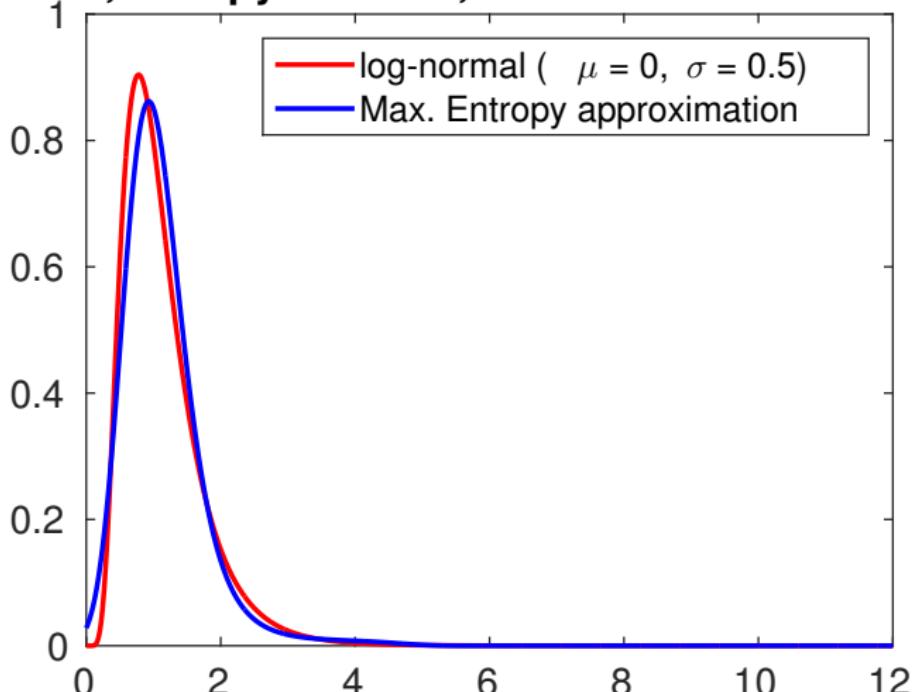
$$R=3, \text{Entropy}=0.84002, \quad \Delta\lambda=5.724e-10$$



### Legendre Moments:

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

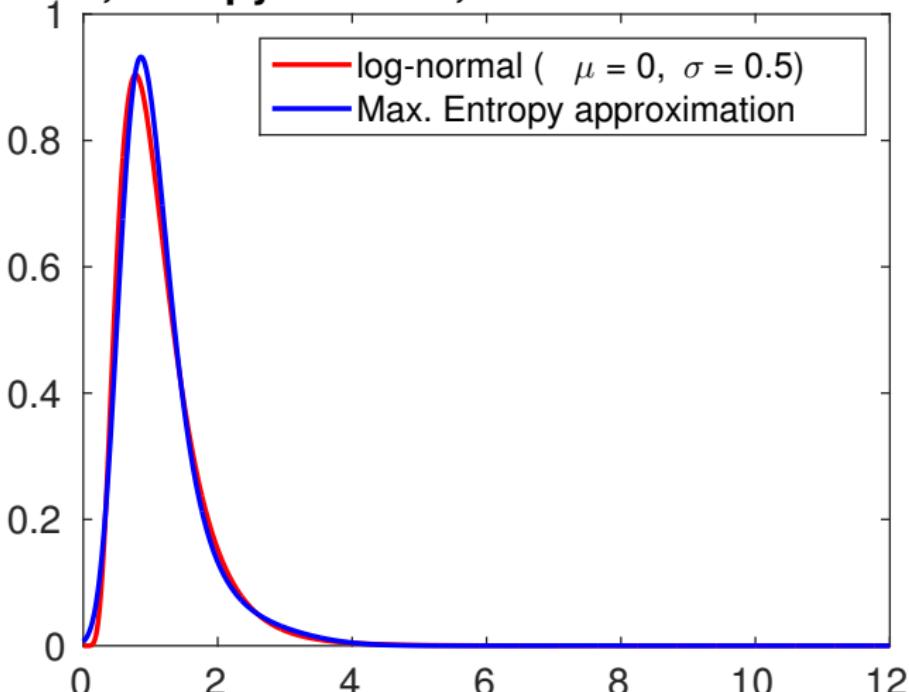
$$R=4, \text{Entropy}=0.76128, \quad \Delta\lambda=6.9085e-10$$



### Legendre Moments:

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

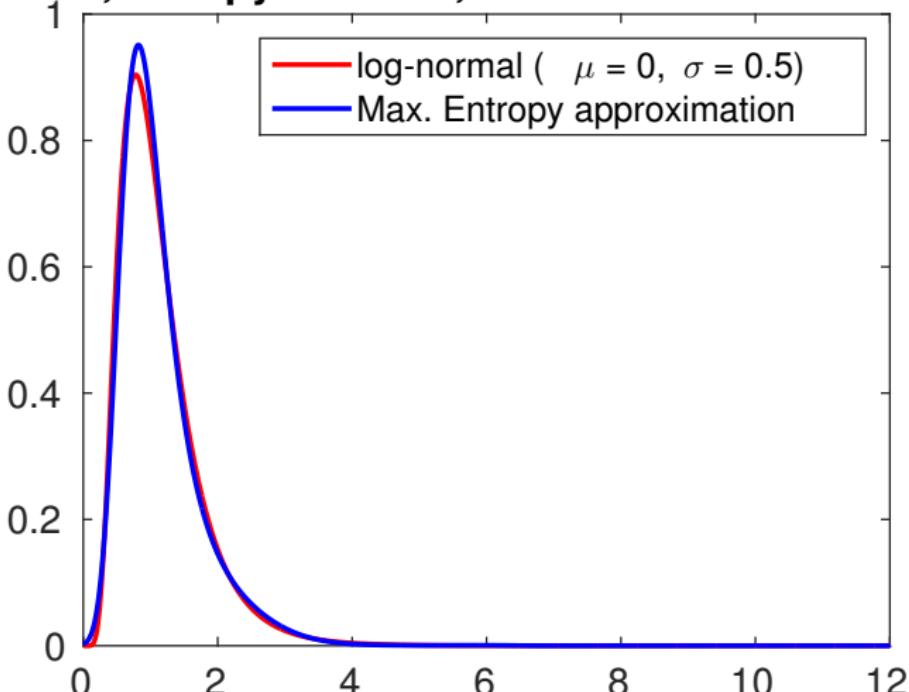
$$R=6, \text{Entropy}=0.73853, \quad \Delta\lambda=2.3278e-10$$



### Legendre Moments:

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

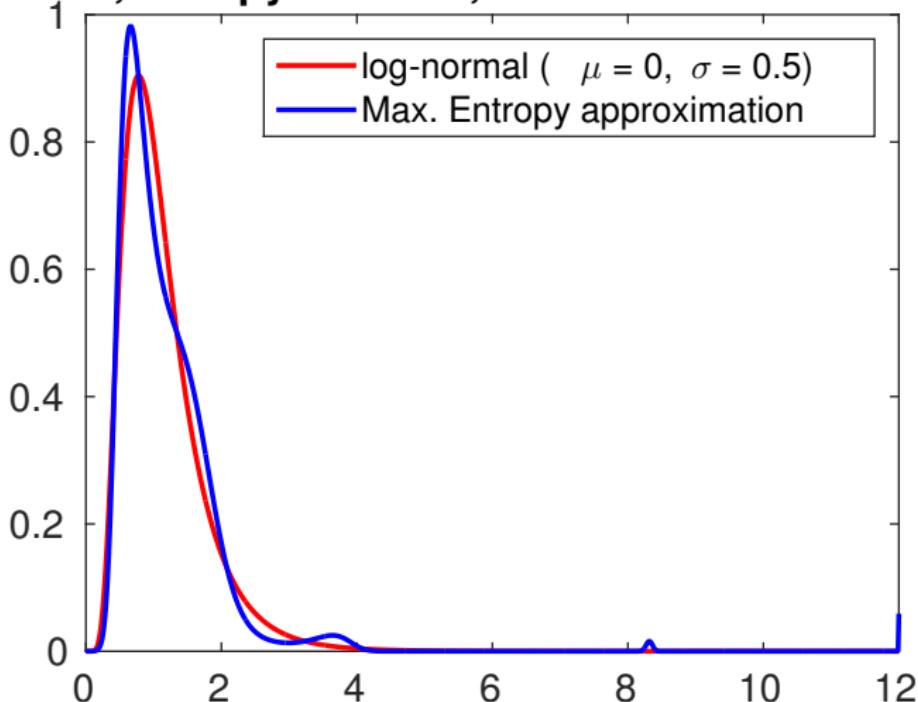
$$R=7, \text{Entropy}=0.73154, \quad \Delta\lambda=7.7511e-10$$



### Legendre Moments:

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

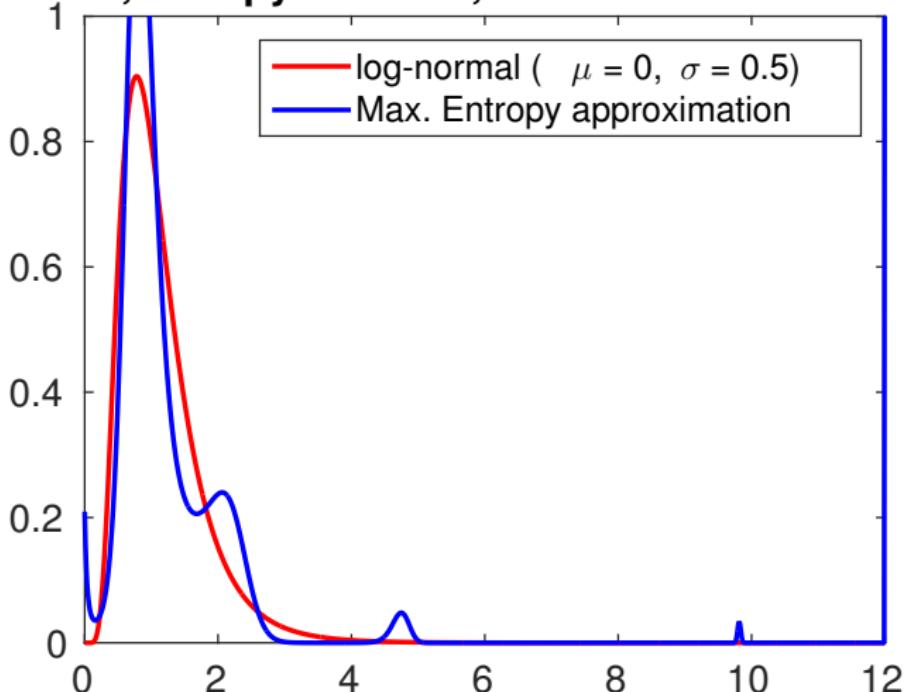
$$R=9, \text{Entropy}=0.70398, \quad \Delta\lambda=1019.0347$$



### Legendre Moments:

- Stable for  $R \leq 8$
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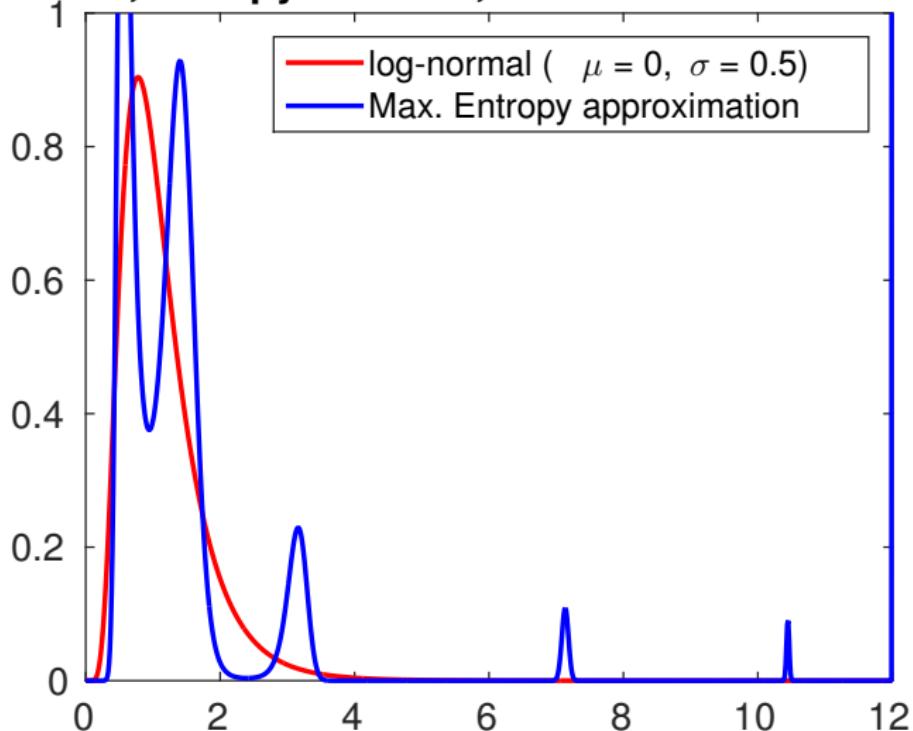
$$R=10, \text{Entropy}=0.71467, \Delta\lambda=1379.7849$$



### Legendre Moments:

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

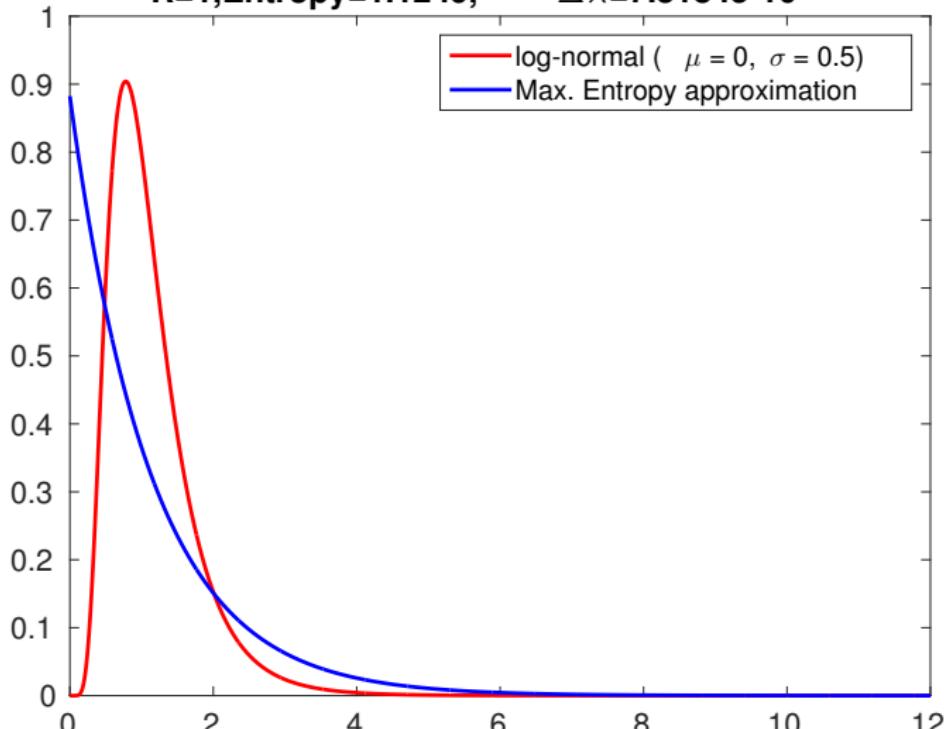
$R=11$ , Entropy = 0.69014,  $\Delta \lambda = 1153.5169$



**Legendre Moments:**

- Stable for  $R \leq 8$
- Entropy is may decrease even when the Newton Method does not converge

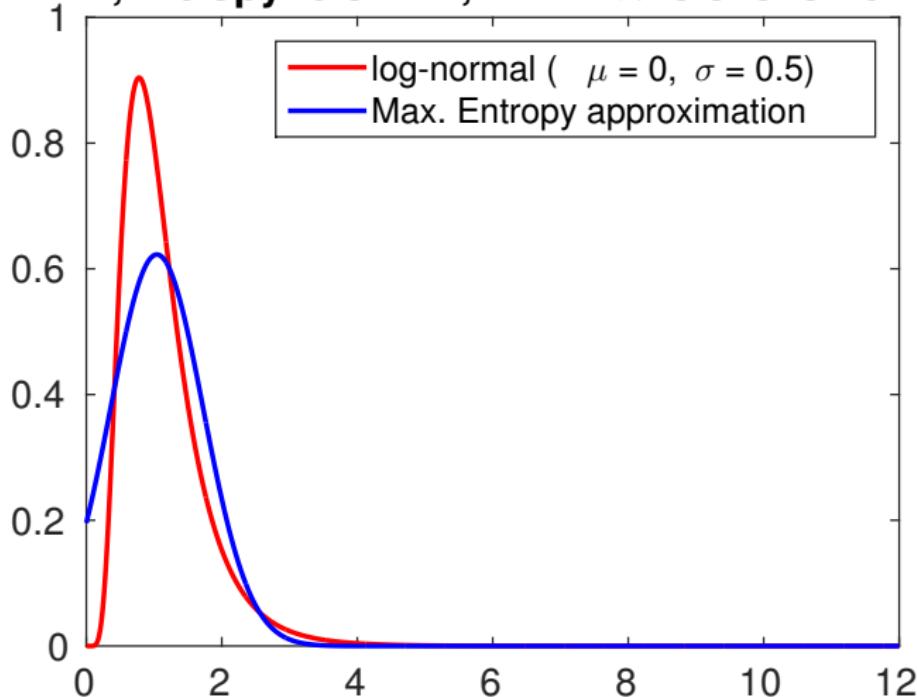
$$R=1, \text{Entropy}=1.1249, \Delta\lambda=7.3184\text{e-}10$$



Monomial Moments:

• Unstable for  $R \geq 5$

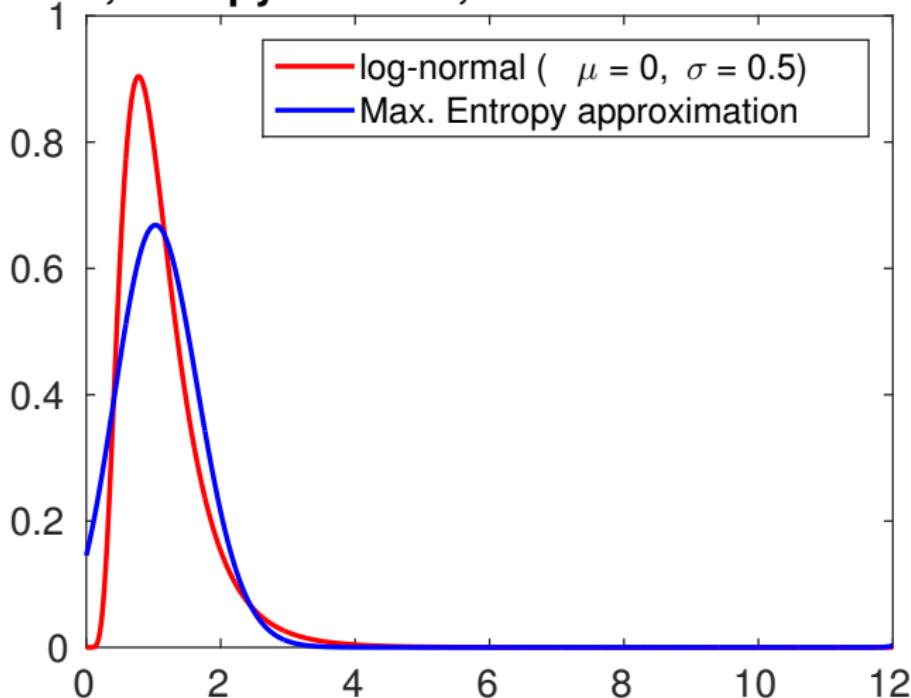
$R=2$ , Entropy=0.87177,  $\Delta \lambda = 8.3457e-10$



Monomial Moments:

● Unstable for  $R \geq 5$

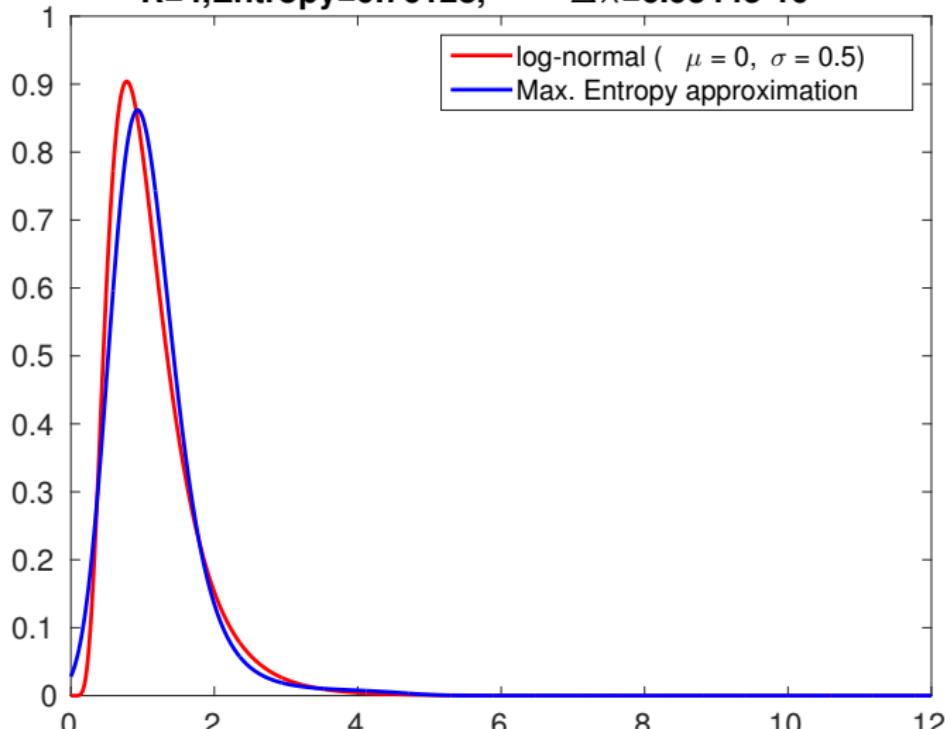
$R=3$ , Entropy = 0.84002,  $\Delta \lambda = 8.6141e-10$



Monomial Moments:

• Unstable for  $R \geq 5$

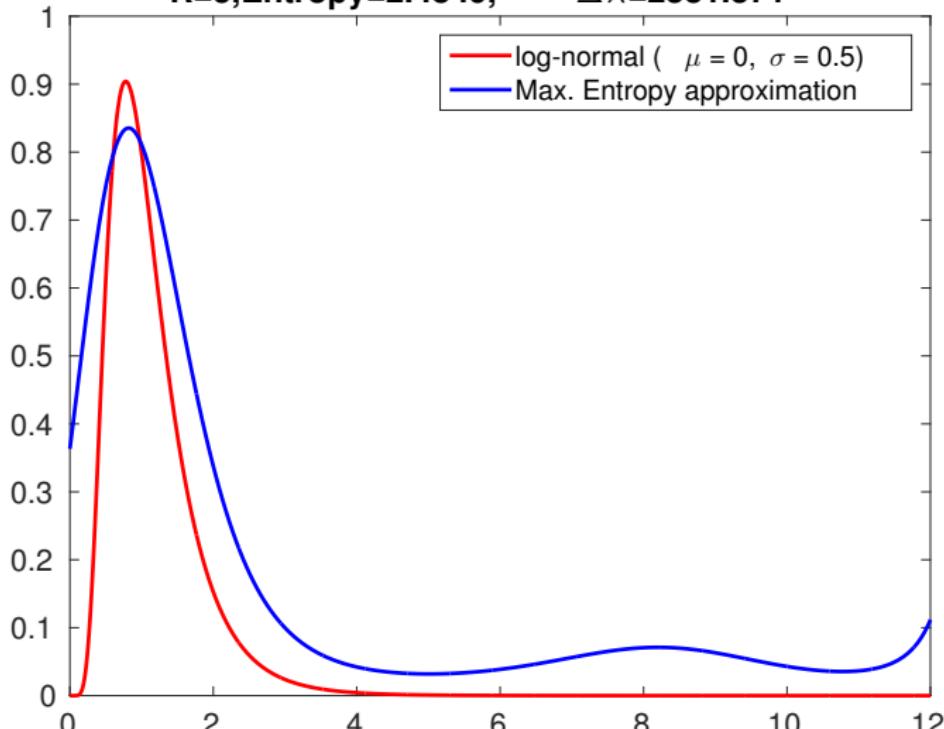
$R=4$ , Entropy = 0.76128,  $\Delta\lambda = 6.9344e-10$



Monomial Moments:

• Unstable for  $R \geq 5$

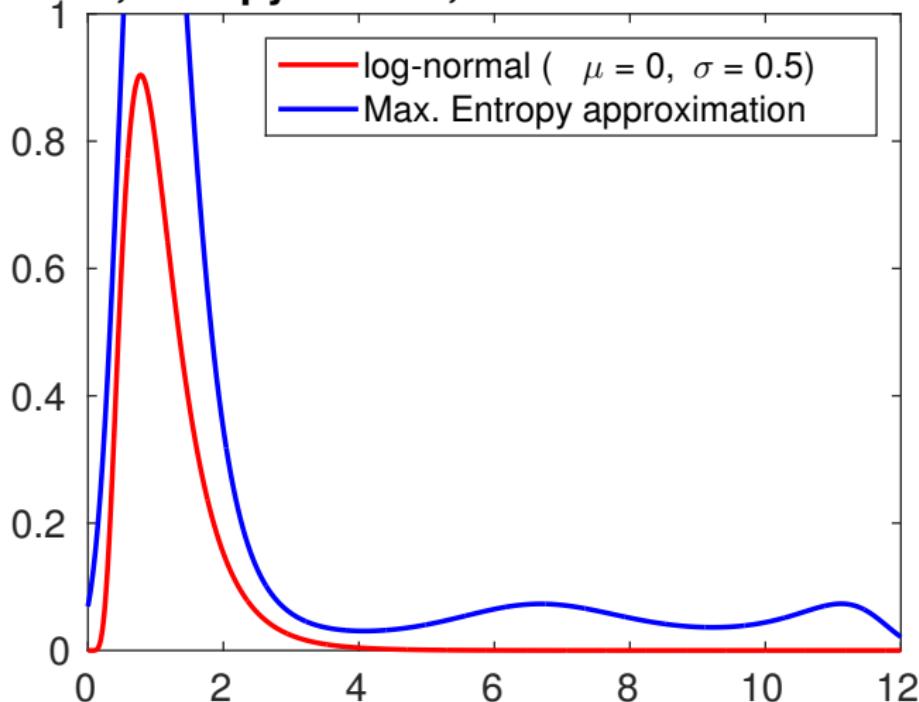
$R=5$ , Entropy = 2.4849,  $\Delta \lambda = 2331.371$



Monomial Moments:

• Unstable for  $R \geq 5$

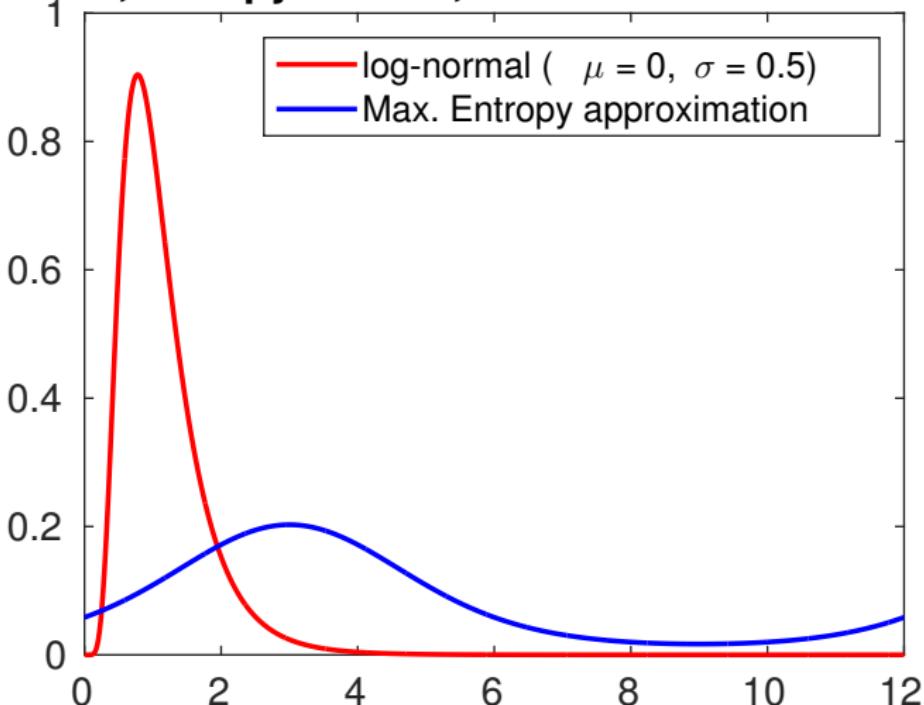
$$R=6, \text{Entropy}=2.4849, \quad \Delta\lambda=17452.8811$$



Monomial Moments:

- Unstable for  $R \geq 5$

$$R=1, \text{Entropy}=2.1851, \quad \Delta\lambda=7.2644e-10$$

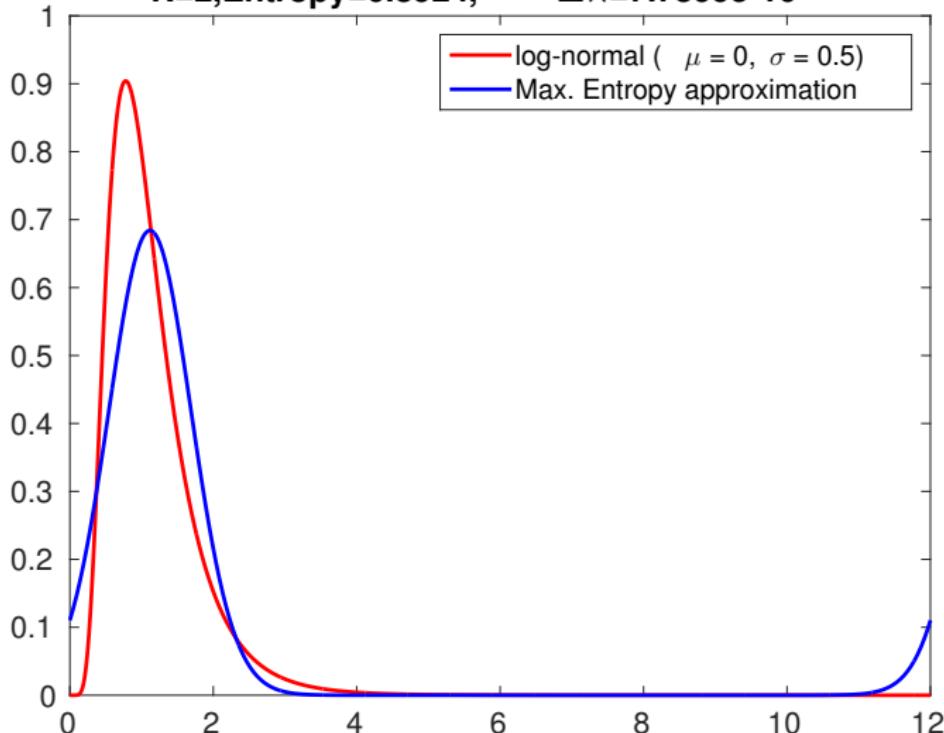


### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

$R=2$ , Entropy = 0.8924,

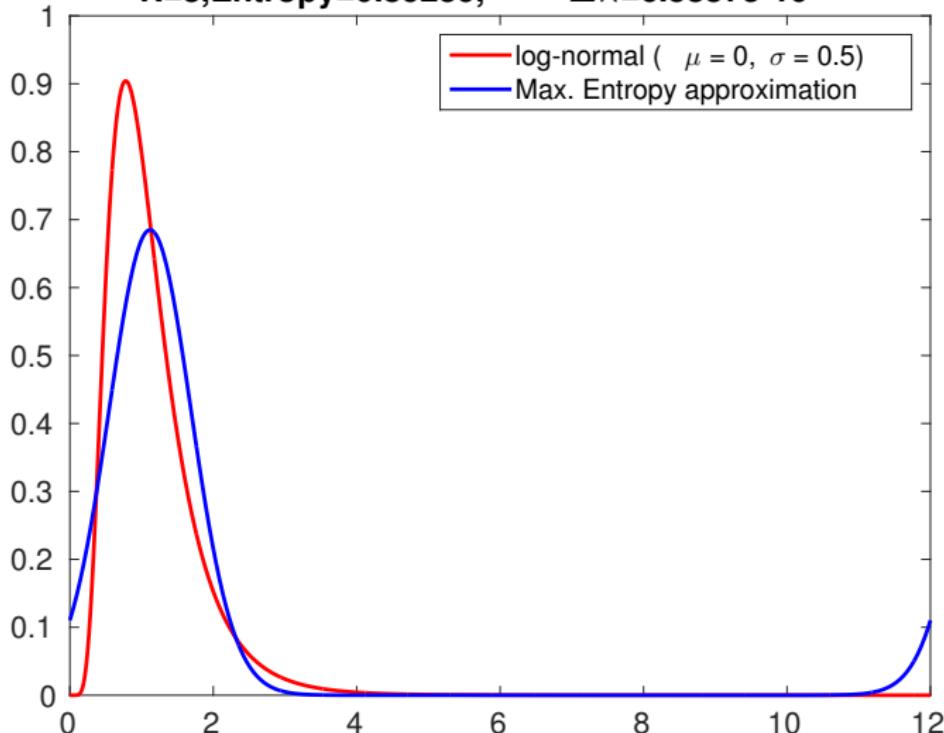
$\Delta \lambda = 7.7399e-10$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

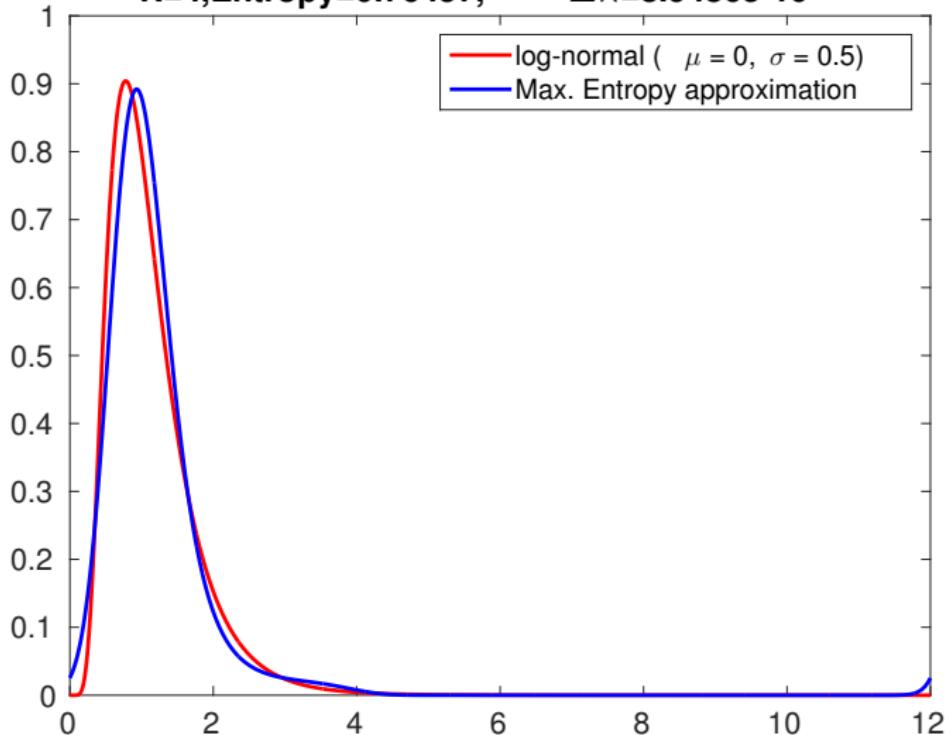
$R=3$ , Entropy = 0.89239,  $\Delta\lambda = 6.5837 \times 10^{-10}$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

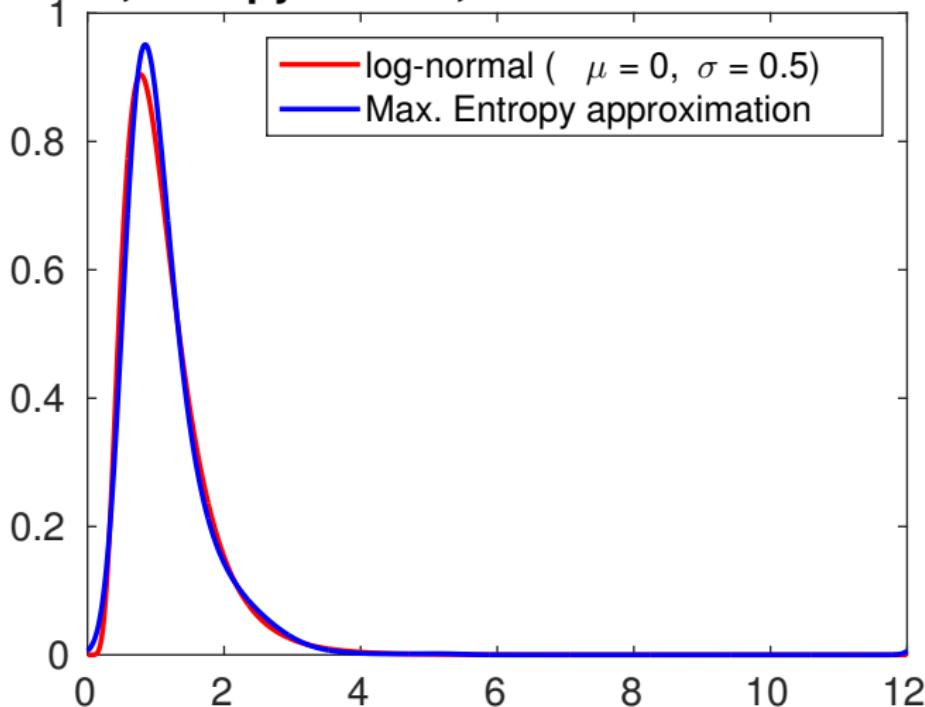
$$R=4, \text{Entropy}=0.76457, \Delta\lambda=8.9486e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

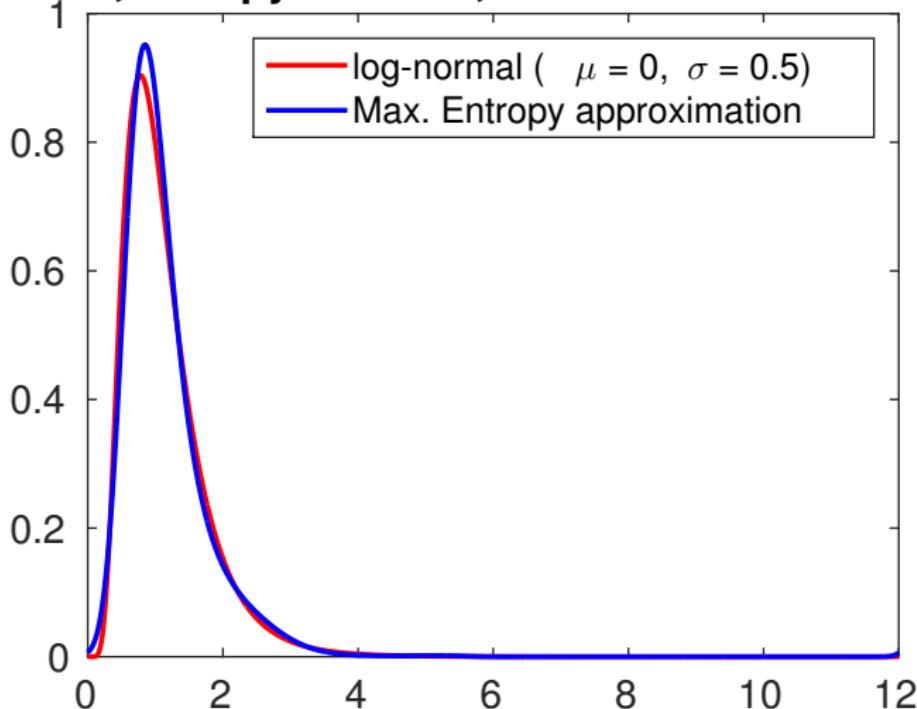
$R=5$ , Entropy = 0.7393,  $\Delta \lambda = 5.7226e-10$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

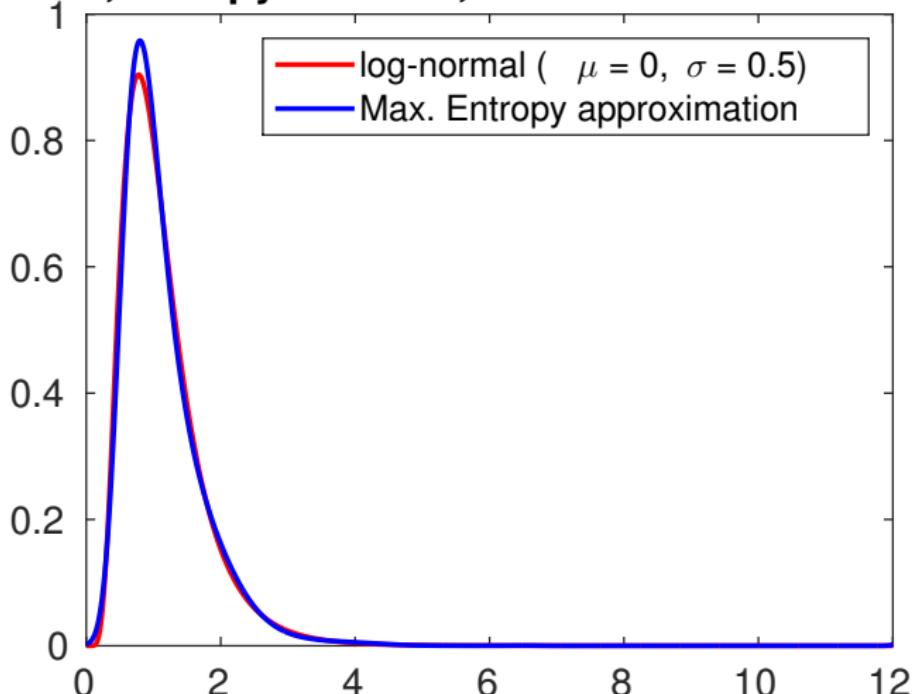
$R=6$ , Entropy = 0.73924,  $\Delta \lambda = 6.765e-10$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

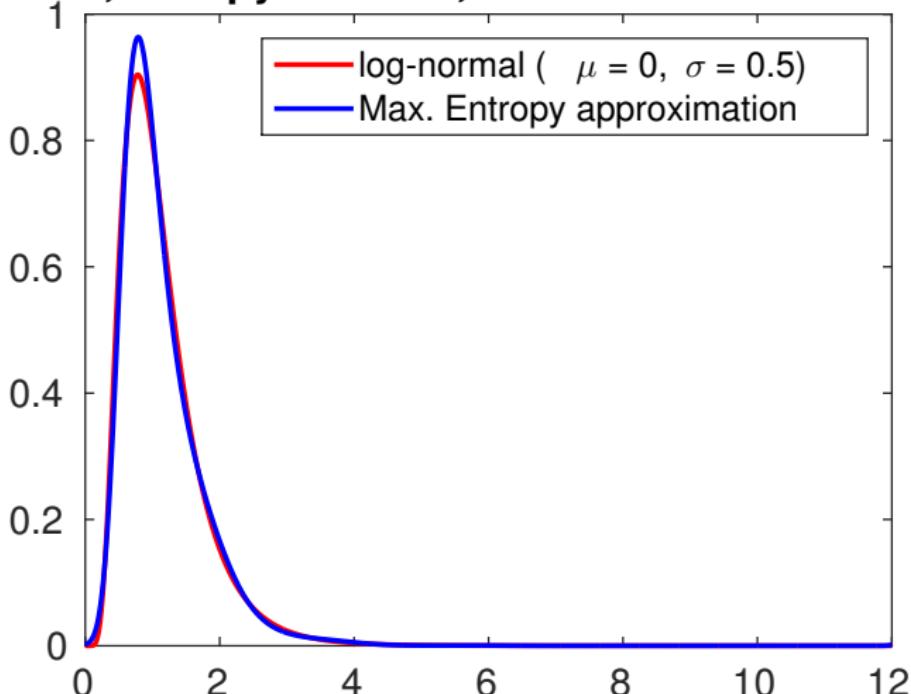
$$R=8, \text{Entropy}=0.73161, \quad \Delta\lambda=8.4948e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

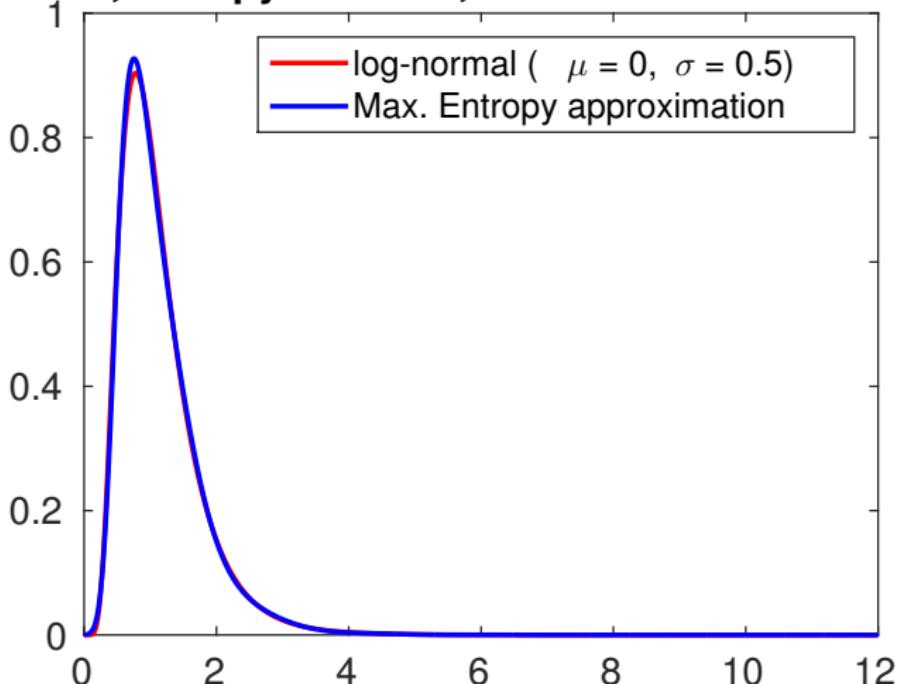
$$R=9, \text{Entropy}=0.72997, \quad \Delta\lambda=9.0555e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

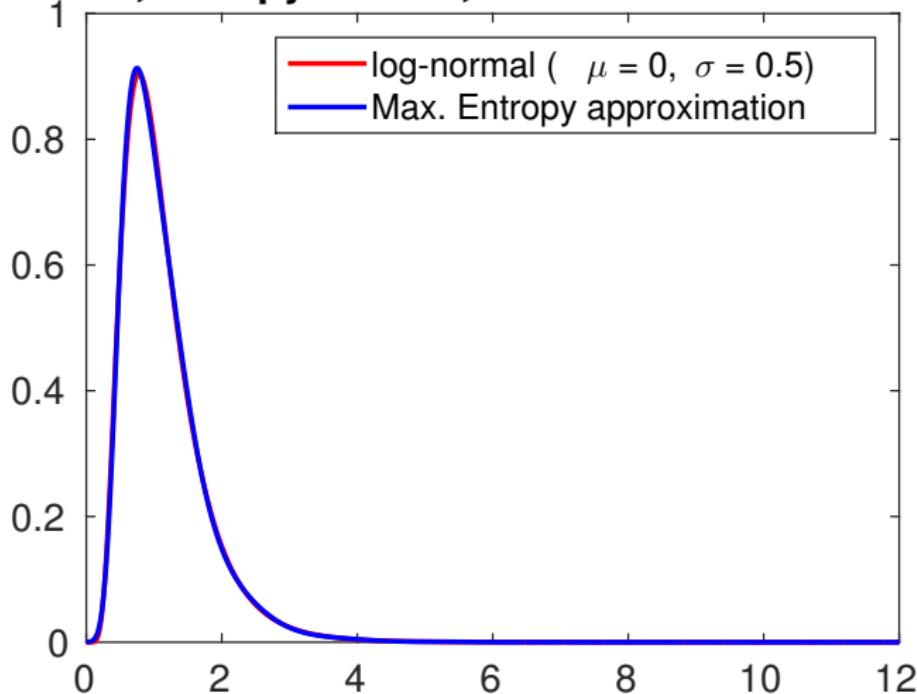
$$R=13, \text{Entropy}=0.72736, \quad \Delta \lambda = 2.9376e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

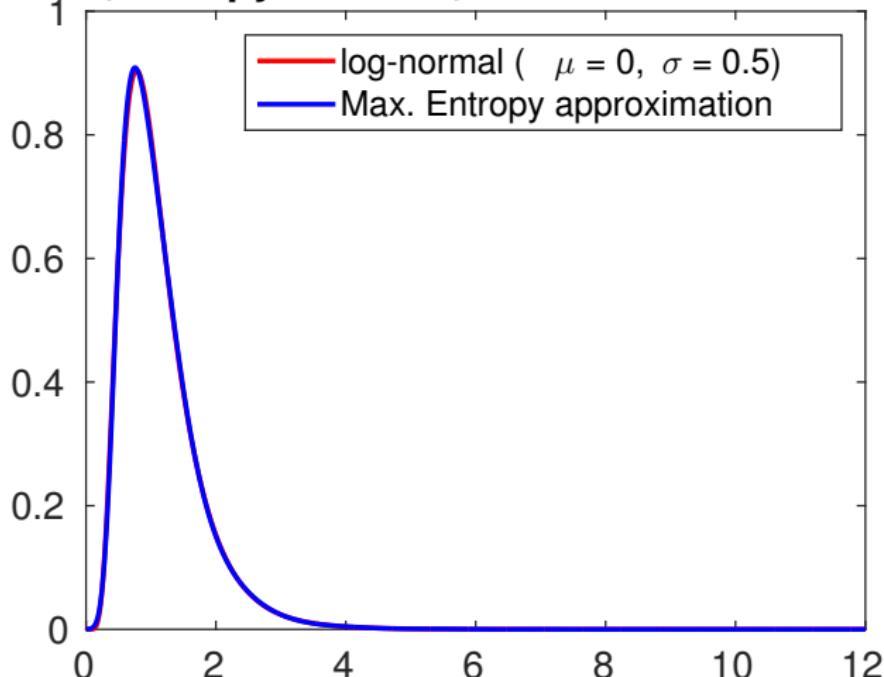
$$R=14, \text{Entropy}=0.7267, \Delta\lambda=5.2192e-11$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

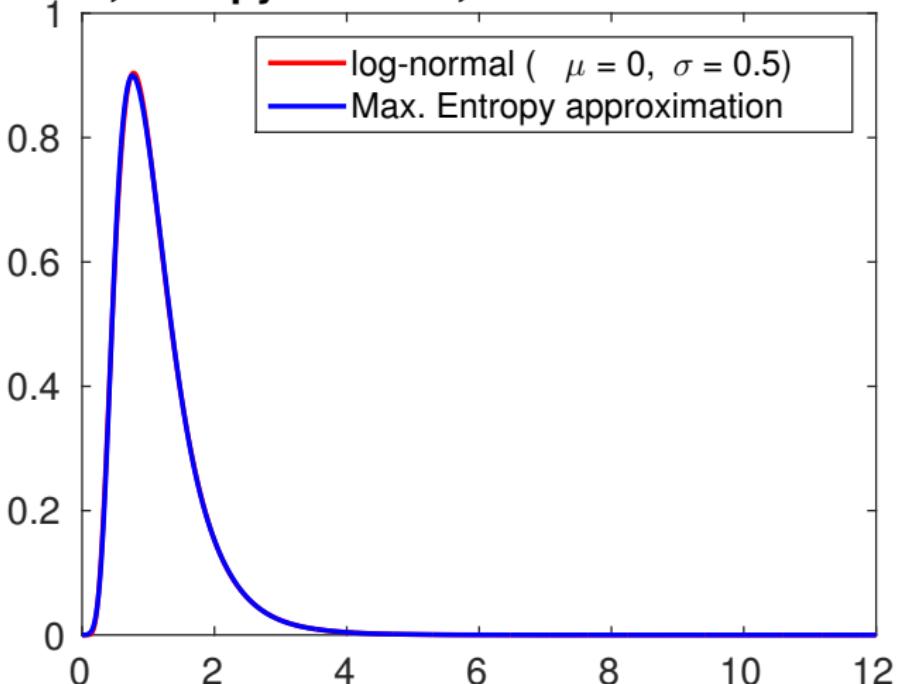
$$R=15, \text{Entropy}=0.72665, \Delta\lambda=4.7698e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

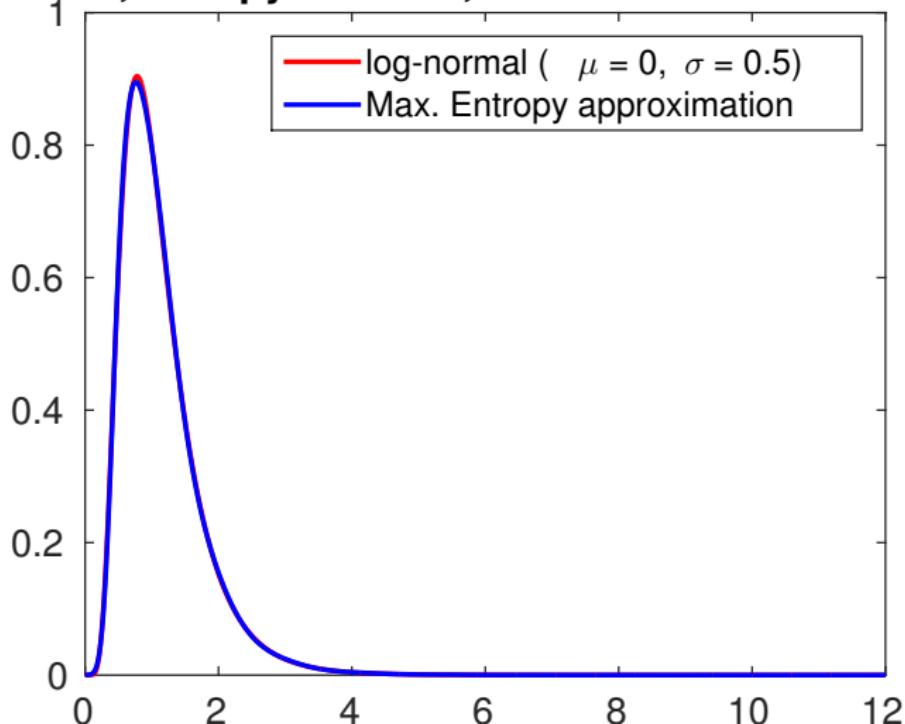
$$R=17, \text{Entropy}=0.72628, \quad \Delta \lambda = 7.7005 \text{e-}10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

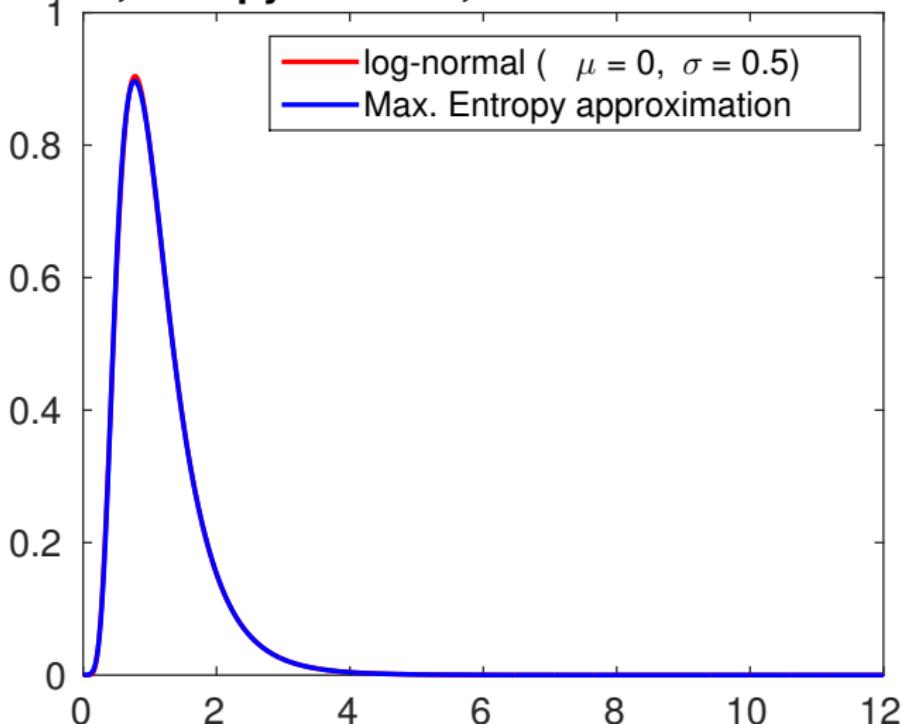
$$R=18, \text{Entropy}=0.72613, \Delta\lambda=7.9606e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

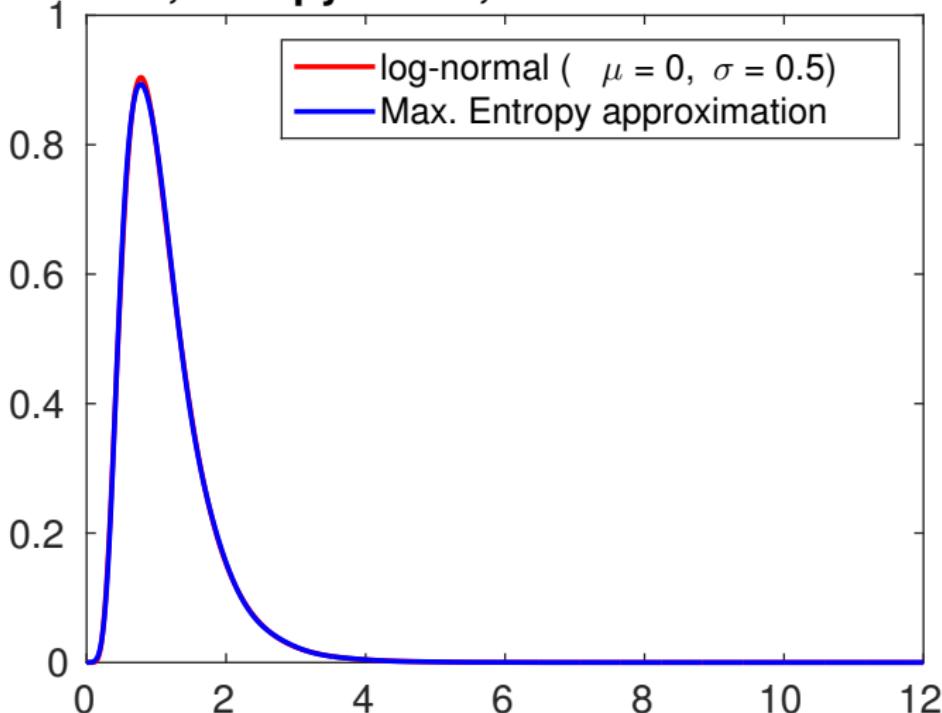
$$R=19, \text{Entropy}=0.72608, \Delta\lambda=6.7018e-10$$



### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

$R=20$ , Entropy = 0.726,  $\Delta \lambda = 8.431e-10$



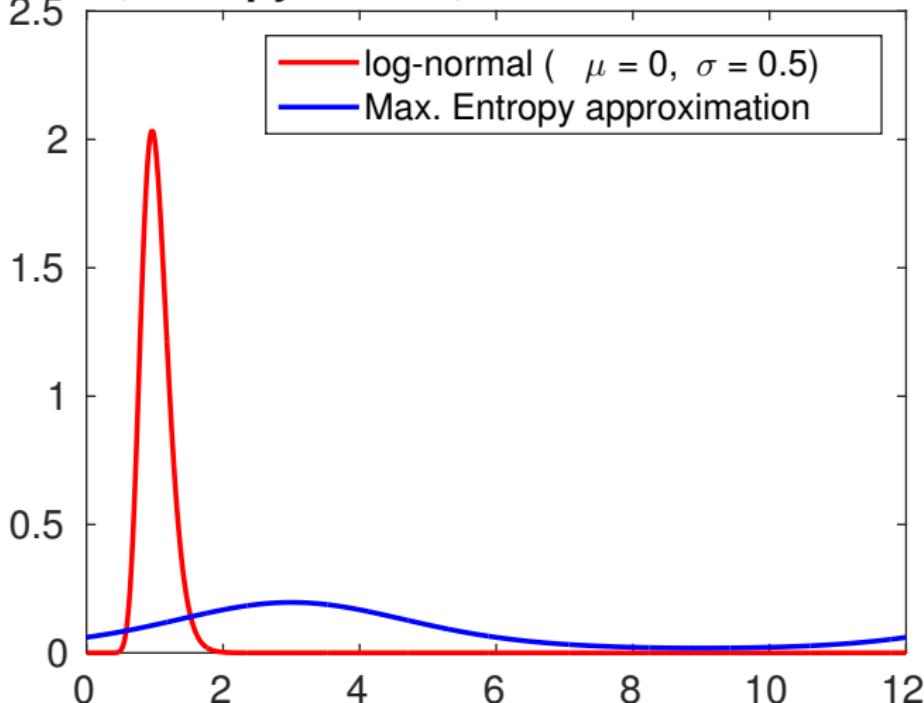
### Fourier Moments:

- Stable
- Entropy is monotonously decreasing

**Breaking convergence for the Fourier basis  
by choosing a more concentrated density!**

e.g. log-normal with  $\mu = 0$ ,  $\sigma = 0.2$

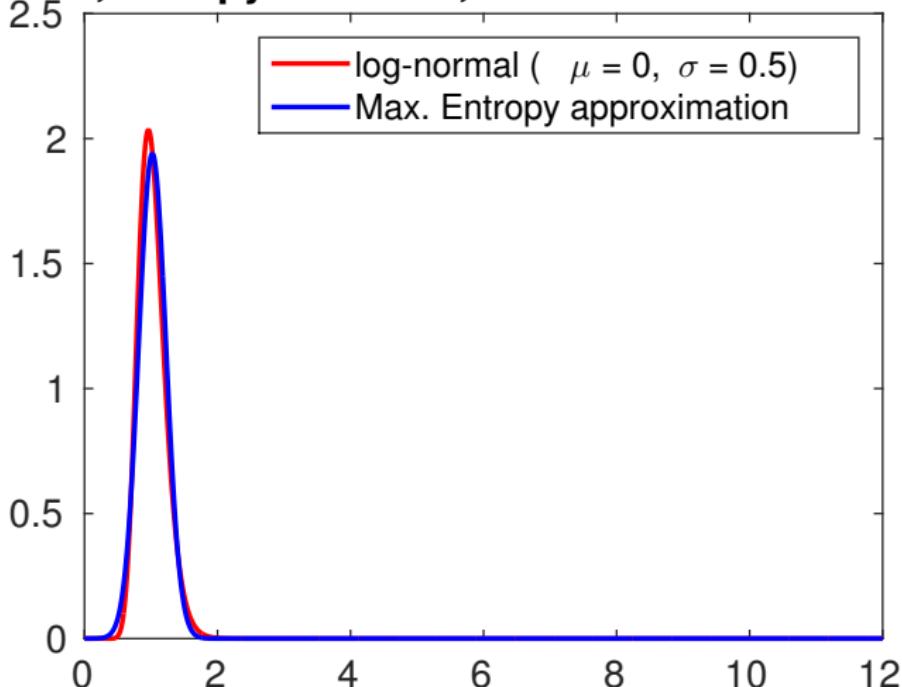
$$R=1, \text{Entropy}=2.2095, \Delta\lambda=6.4032e-10$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

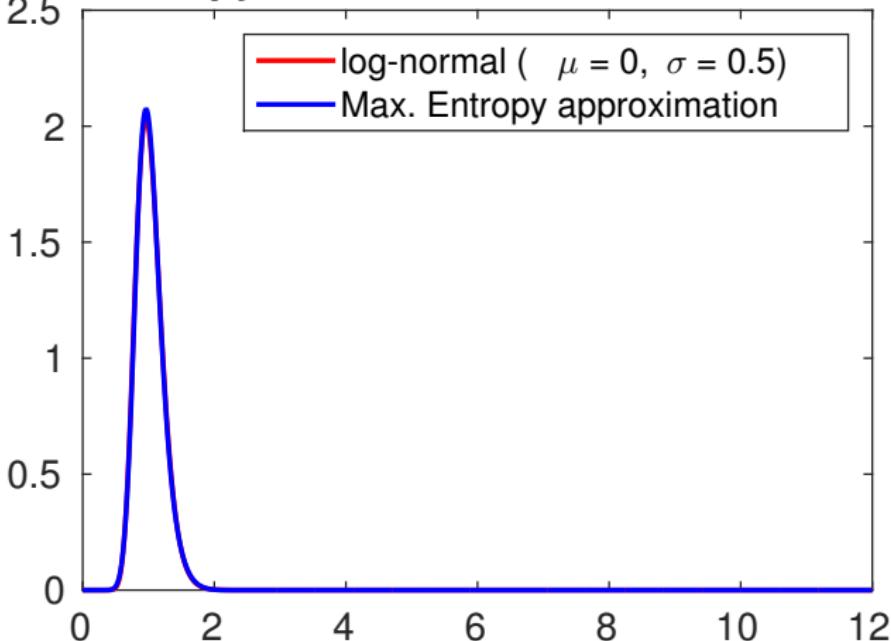
$$R=2, \text{Entropy}=-0.16084, \quad \Delta \lambda = 6.0029e-10$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

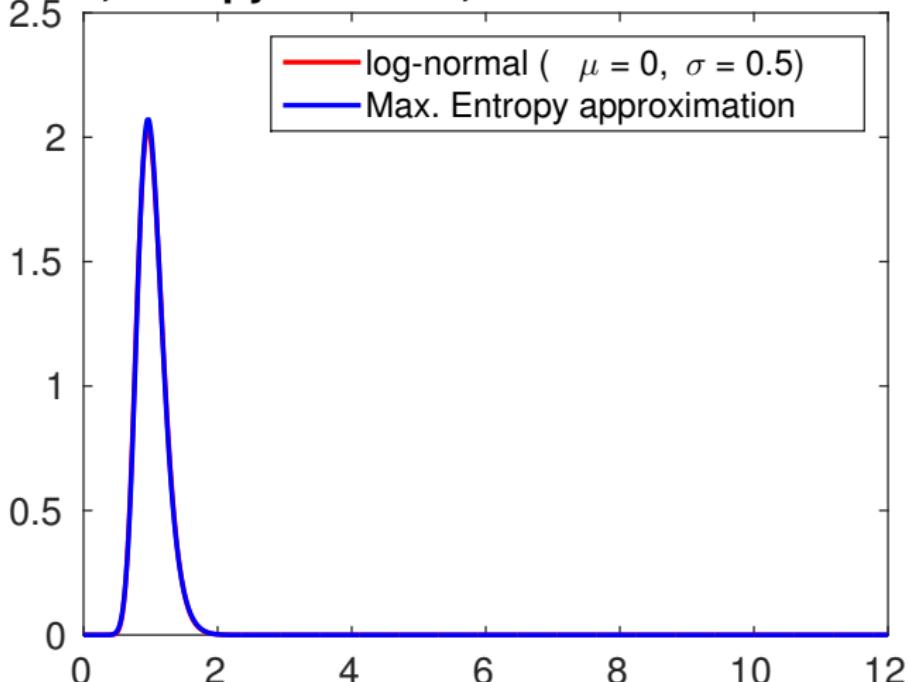
$$R=3, \text{Entropy}=-0.19001, \Delta\lambda=4.9765e-11$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

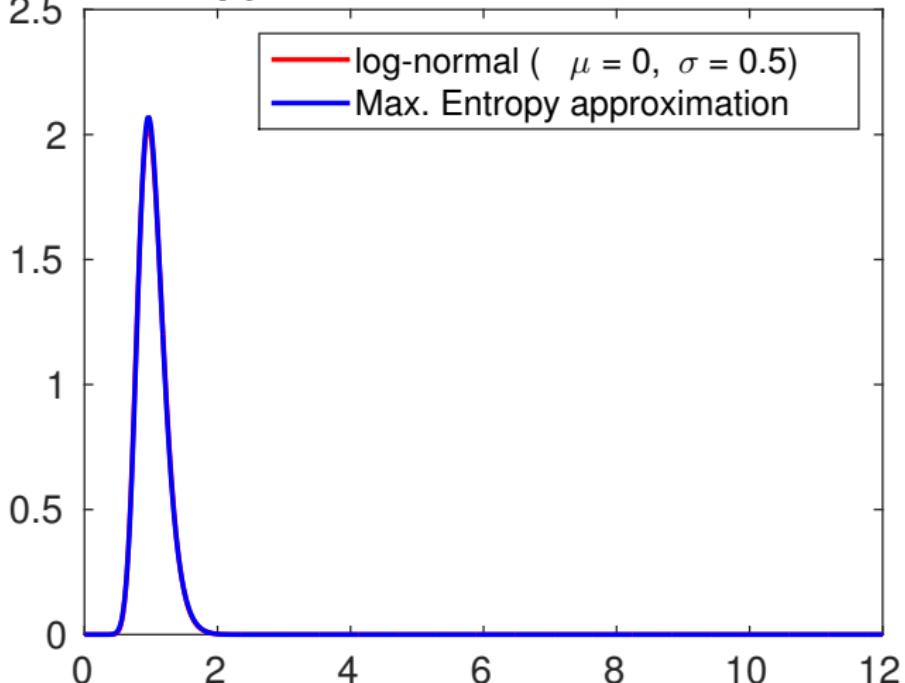
$$R=4, \text{Entropy}=-0.19002, \Delta\lambda=7.8862e-10$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

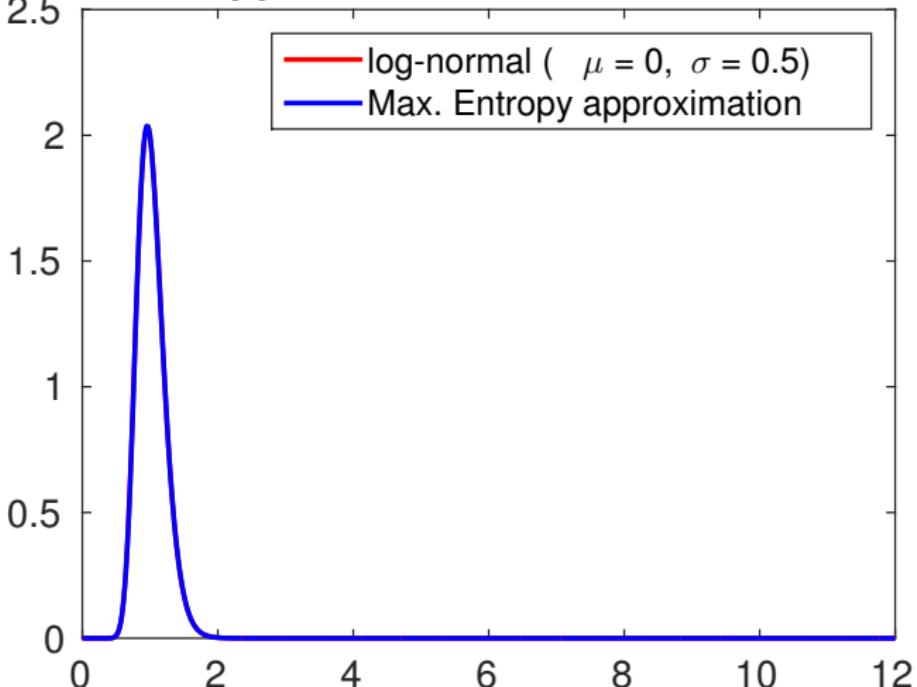
$$R=5, \text{Entropy}=-0.19012, \Delta\lambda=8.2299e-10$$



**Fourier Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

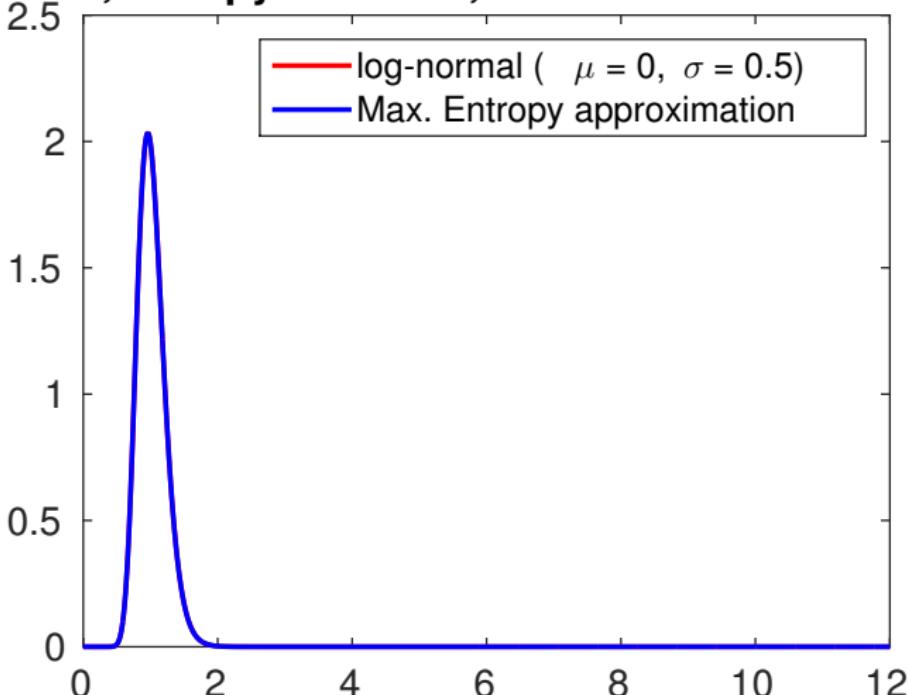
$$R=6, \text{Entropy}=-0.19048, \quad \Delta\lambda=1071.3155$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

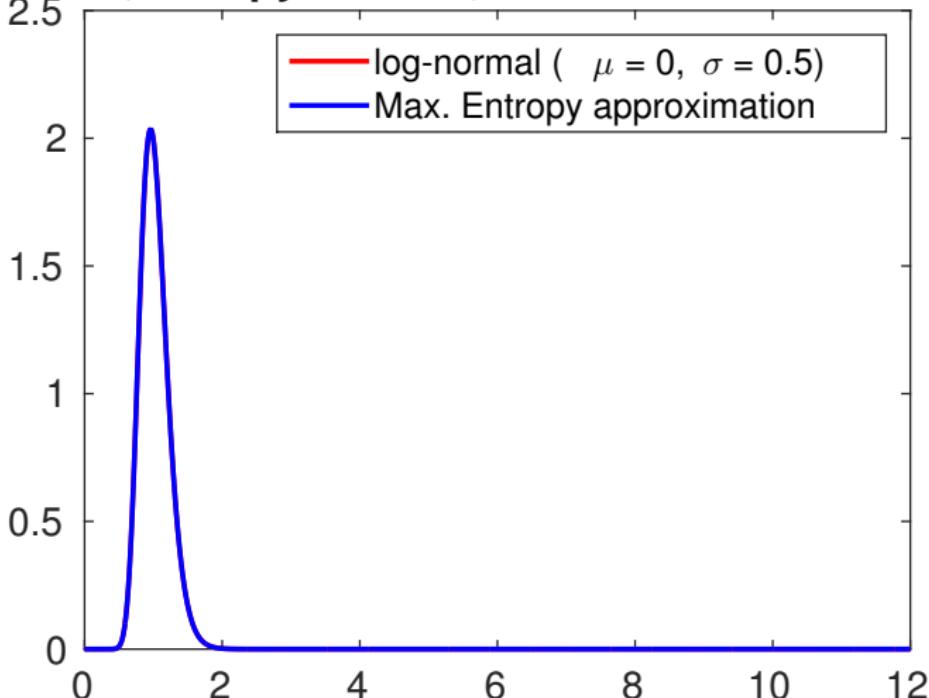
$$R=8, \text{Entropy}=-0.19054, \quad \Delta\lambda=1011.0382$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

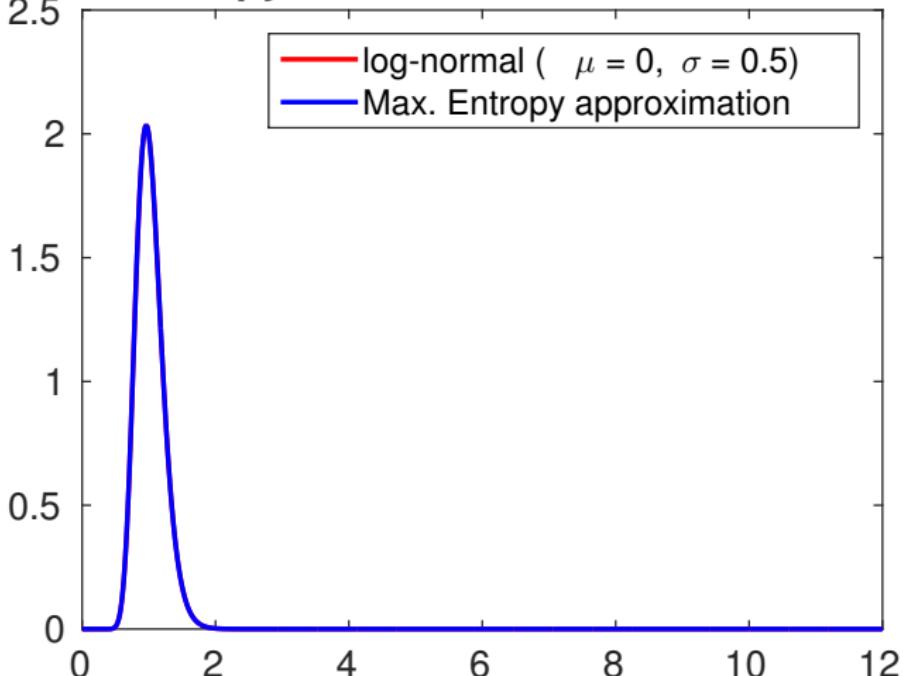
$$R=9, \text{Entropy}=-0.1905, \quad \Delta\lambda=2342.9086$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

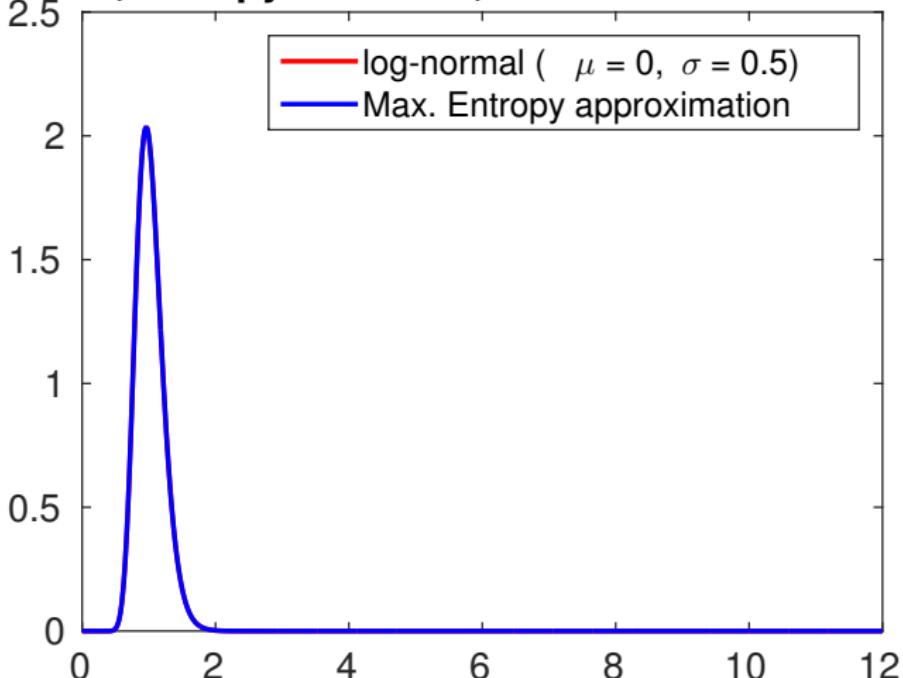
$$R=13, \text{Entropy}=-0.19055, \quad \Delta\lambda=1117.2321$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

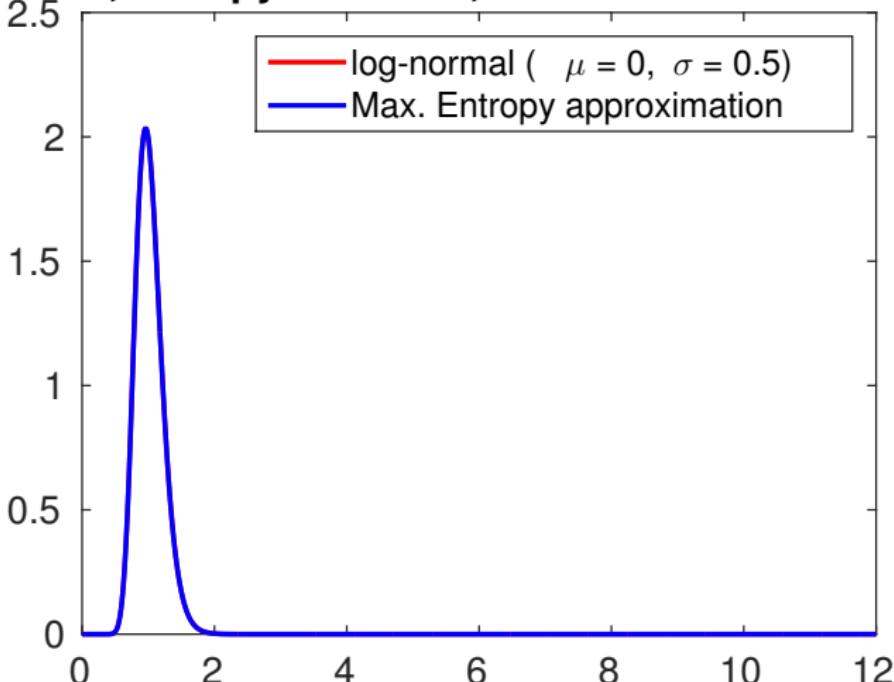
$$R=14, \text{Entropy}=-0.19055, \quad \Delta\lambda=1307.7152$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

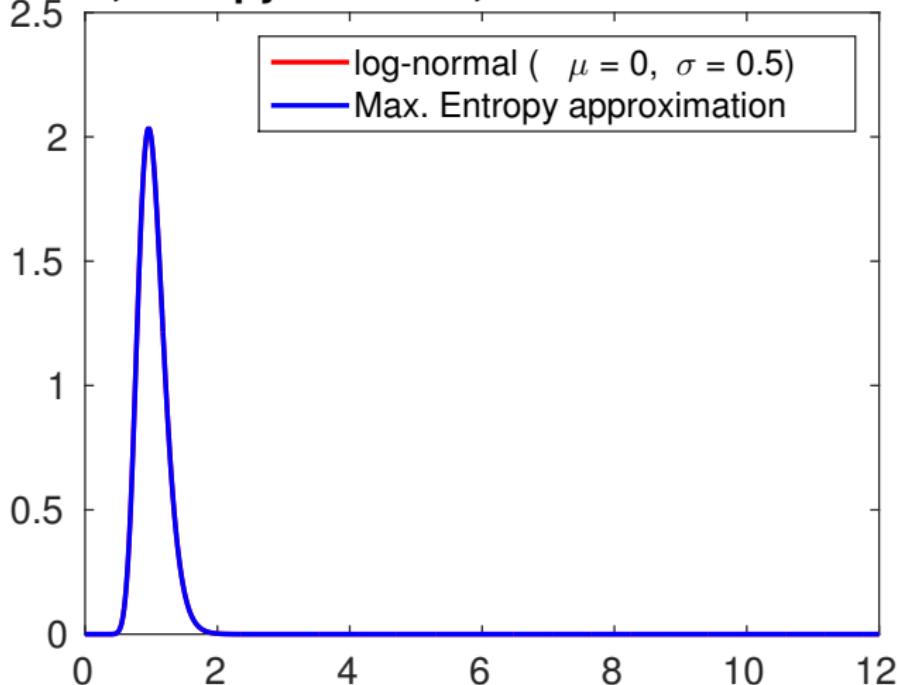
$$R=15, \text{Entropy}=-0.19055, \Delta\lambda=7575.1711$$



**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

$$R=16, \text{Entropy}=-0.19055, \Delta\lambda=1283.0868$$



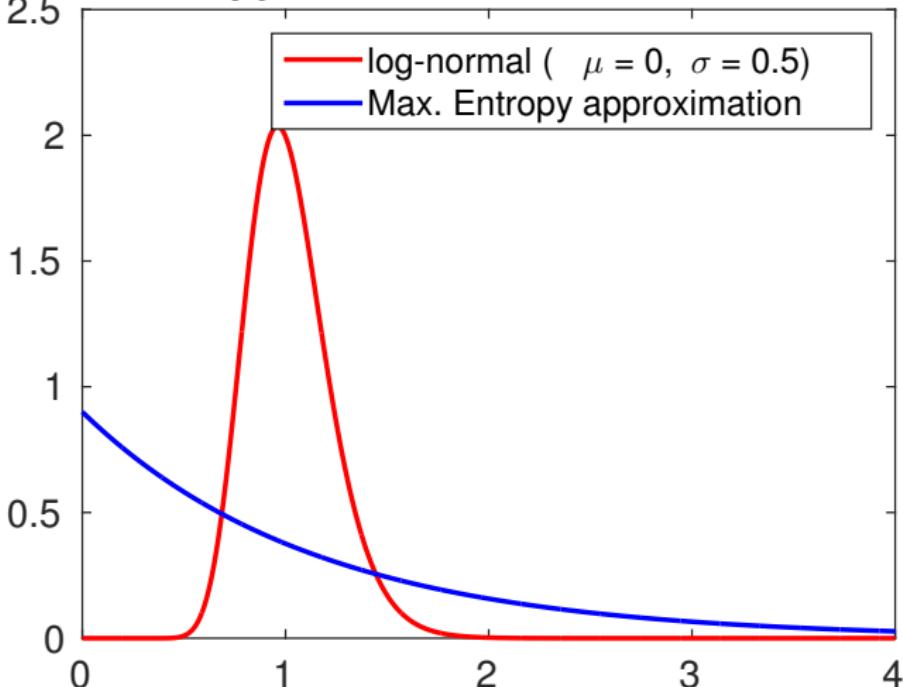
**Fourier Moments ( $\sigma = 0.2, [a, b] = [0, 12]$ ):**

- Remain stable even without convergence!
- Entropy is still monotonously decreasing!

**Regain stability of the Legendre basis  
by choosing a smaller approximation interval!**

e.g.  $[a, b] = [0, 4]$

$$R=1, \text{Entropy}=0.99553, \quad \Delta \lambda = 5.2639e-10$$

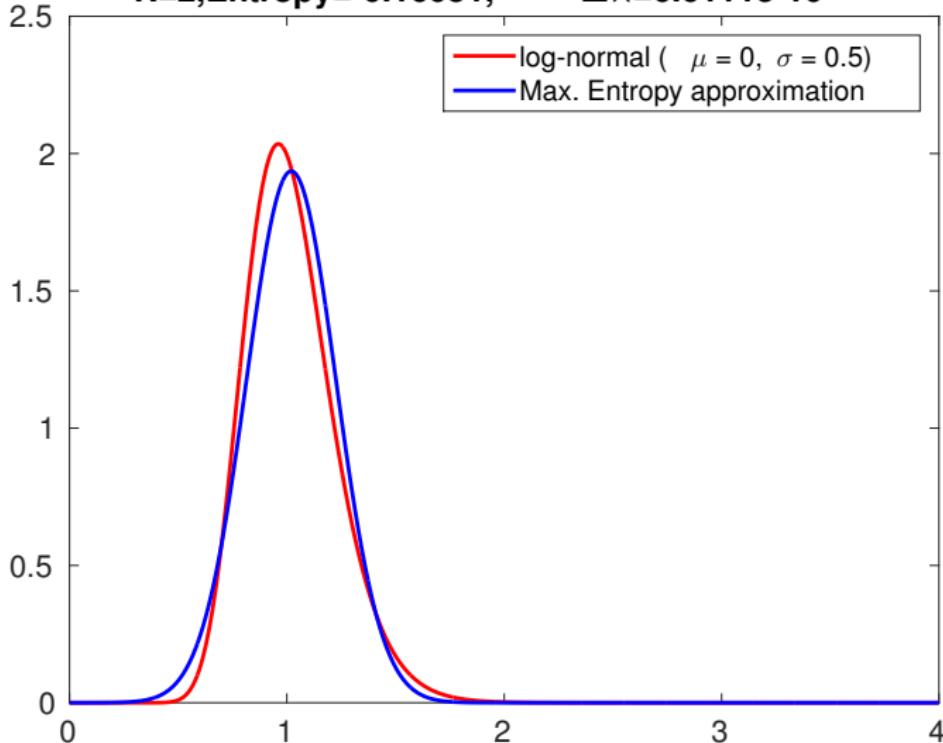


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=2, \text{Entropy}=-0.16051,$

$\Delta\lambda=6.6111e-10$

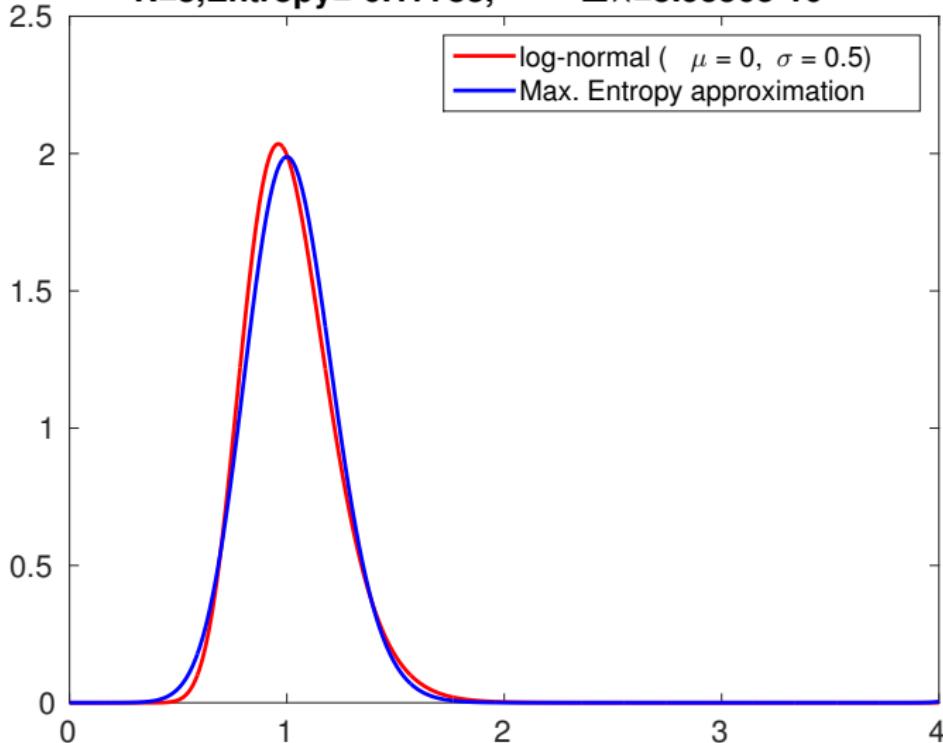


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=3, \text{Entropy}=-0.17783,$

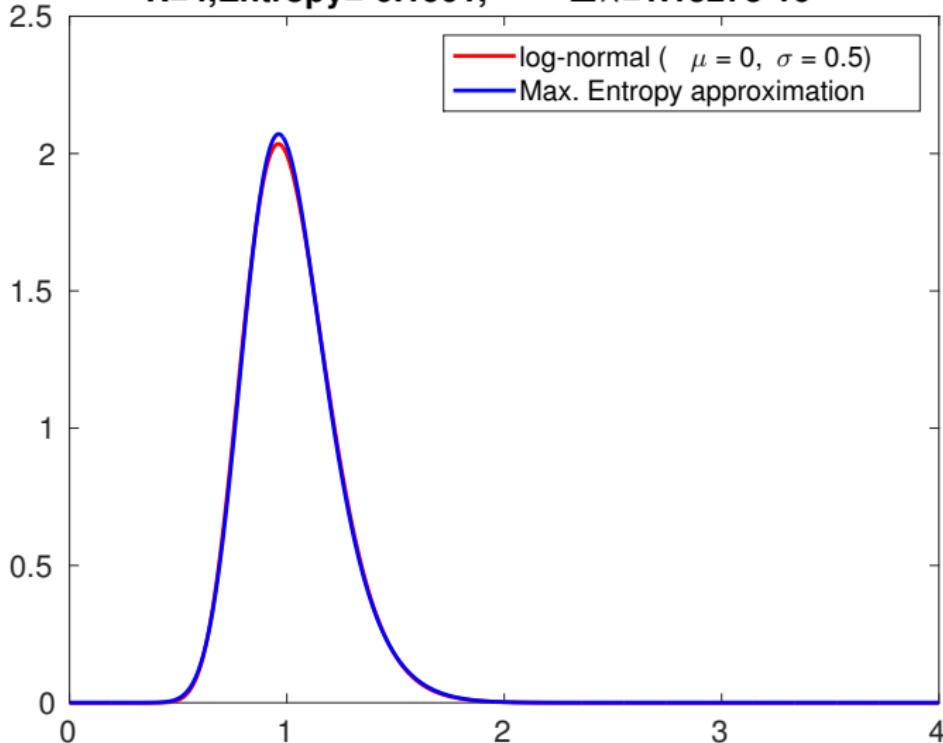
$\Delta\lambda=5.9556e-10$



**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

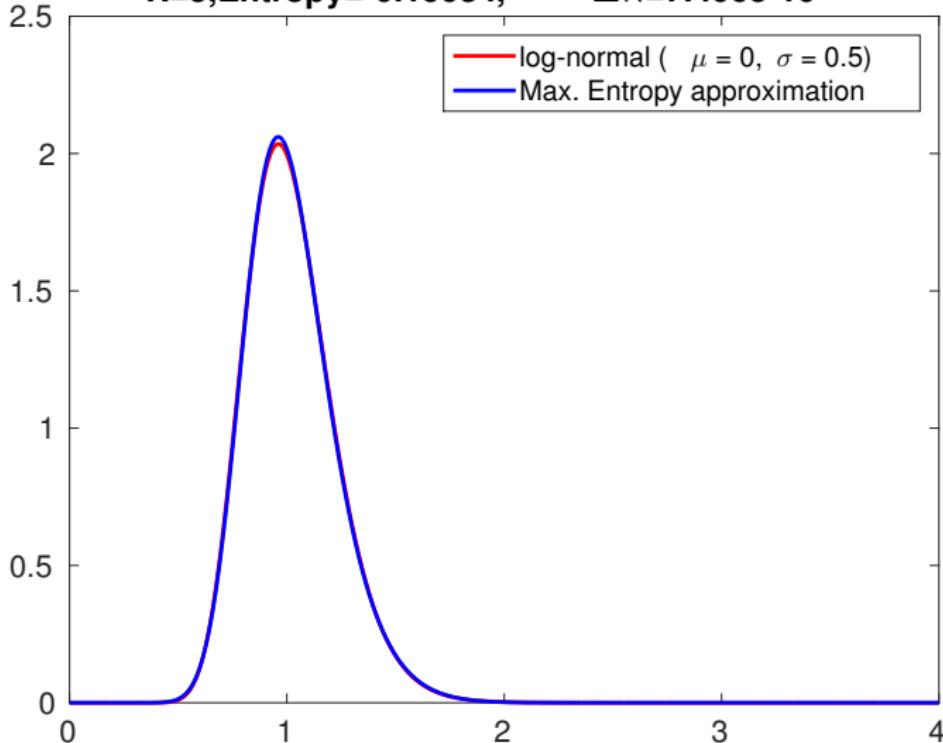
$R=4$ , Entropy = -0.1901,  $\Delta\lambda = 1.1527e-10$



**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=5, \text{Entropy}=-0.19034, \Delta\lambda=7.406e-10$

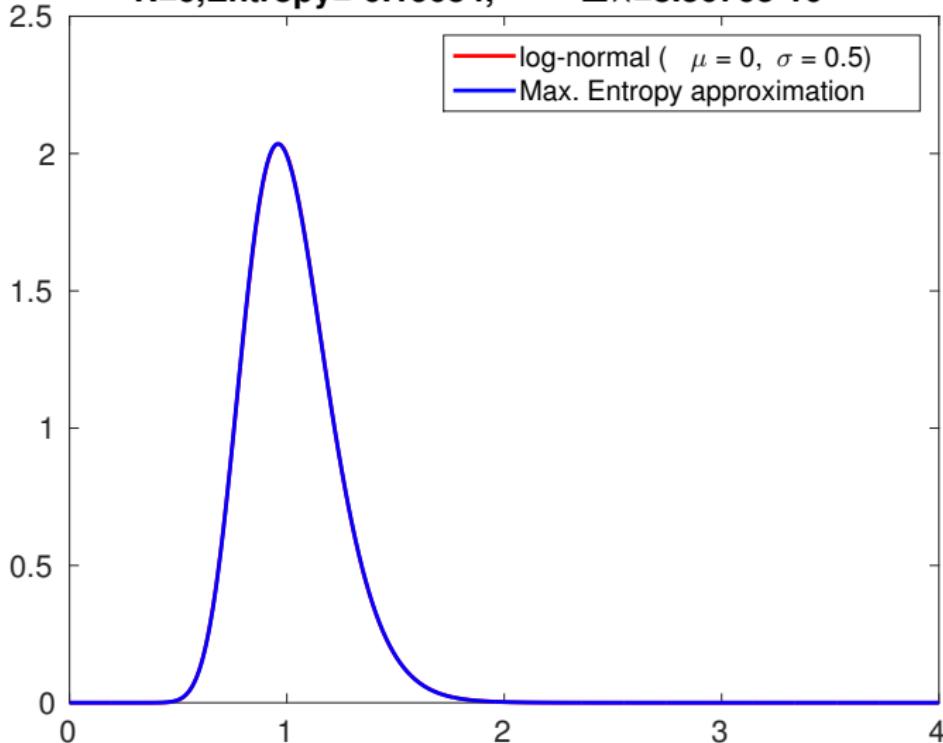


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=6, \text{Entropy}=-0.19054,$

$\Delta\lambda=8.5076e-10$

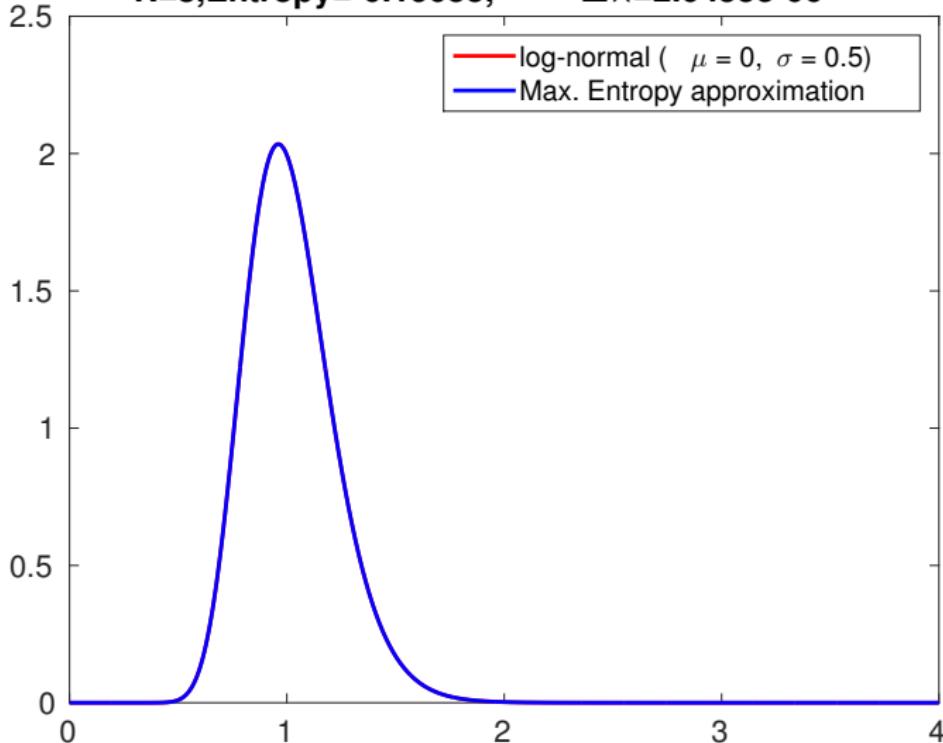


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=8, \text{Entropy}=-0.19055,$

$\Delta\lambda=2.0455e-06$

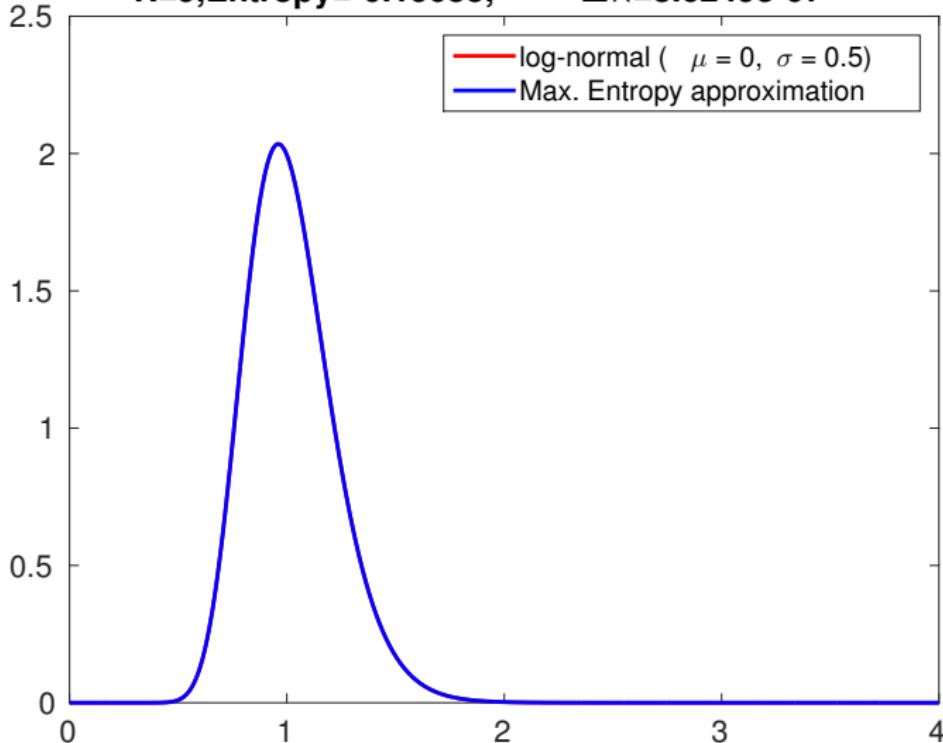


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=9, \text{Entropy}=-0.19055,$

$\Delta\lambda=3.6249e-07$

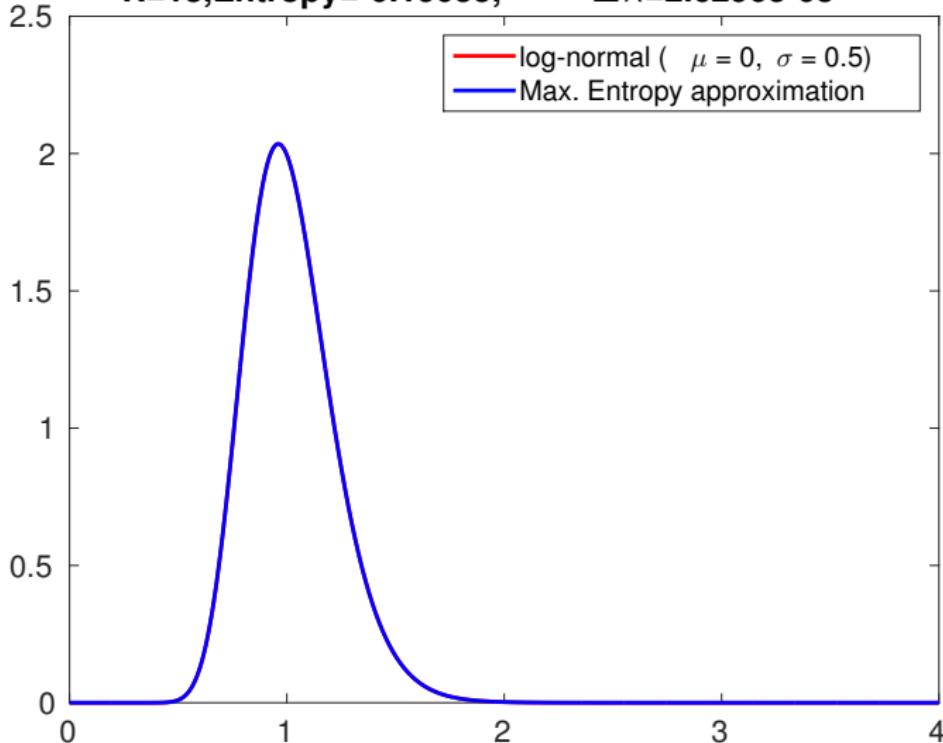


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=13$ , Entropy = -0.19055,

$\Delta \lambda = 2.6296e-05$

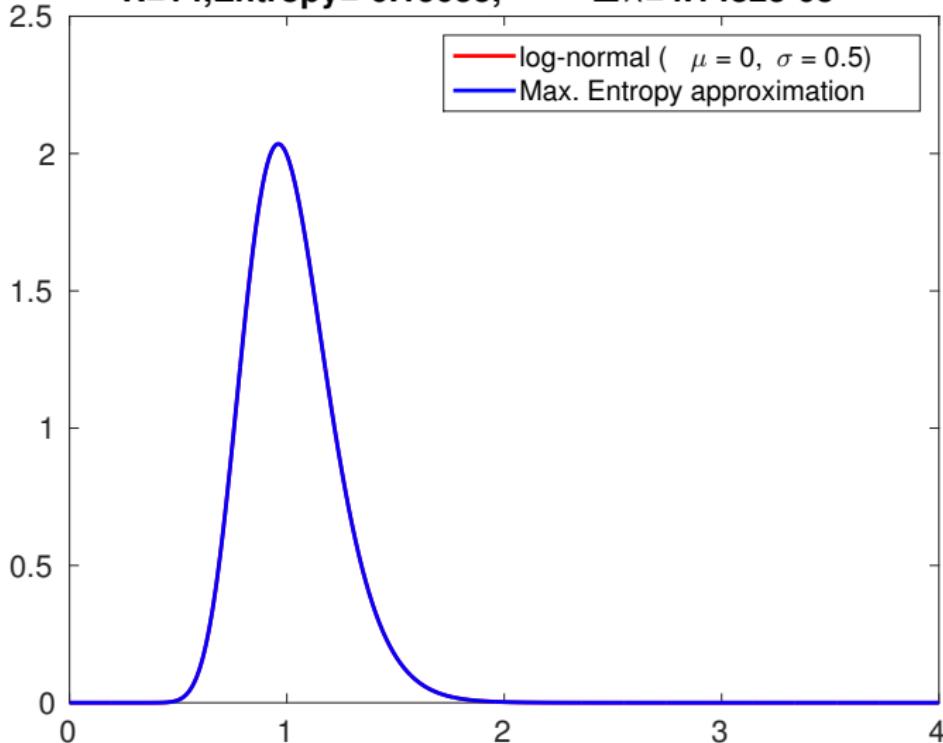


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=14, \text{Entropy}=-0.19055,$

$\Delta\lambda=4.1482e-05$

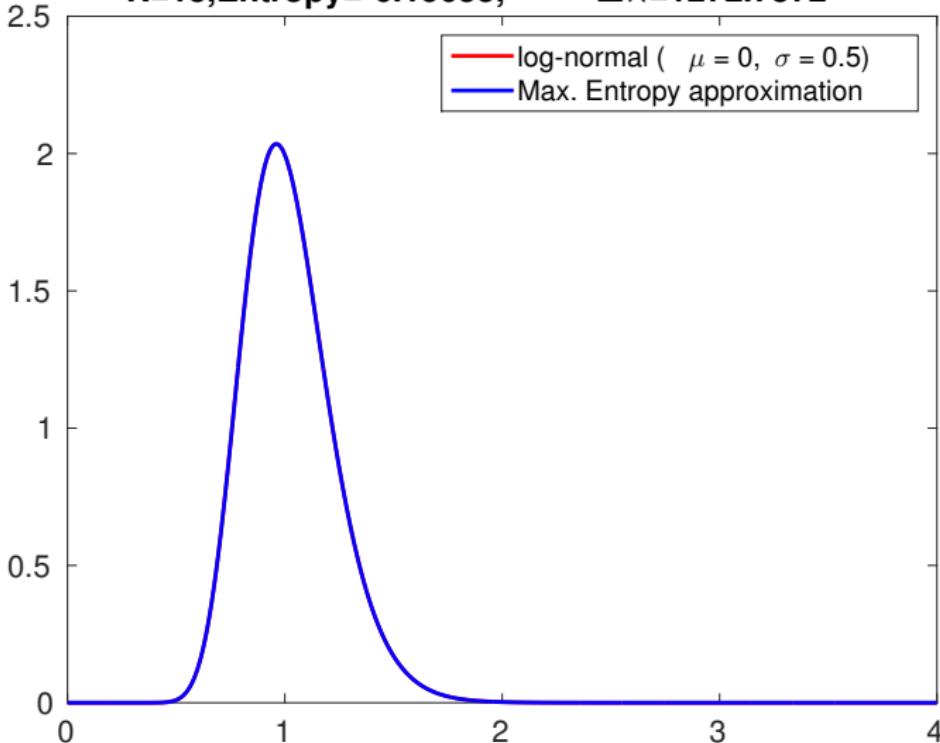


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=15, \text{Entropy}=-0.19055,$

$\Delta\lambda=1272.7572$

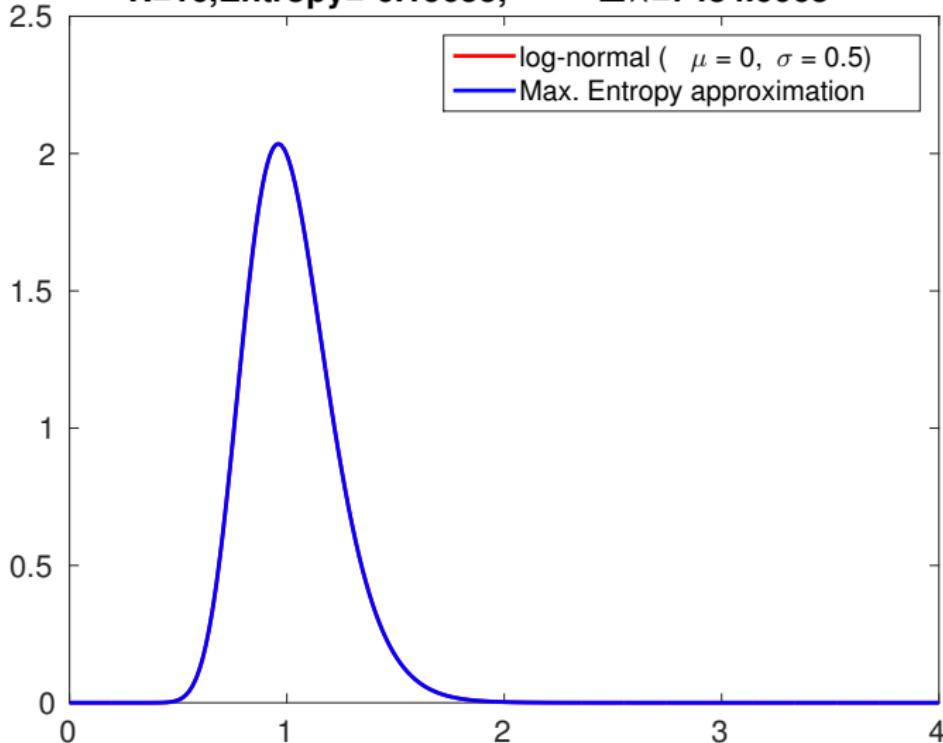


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=16, \text{Entropy}=-0.19055,$

$\Delta\lambda=7434.6063$

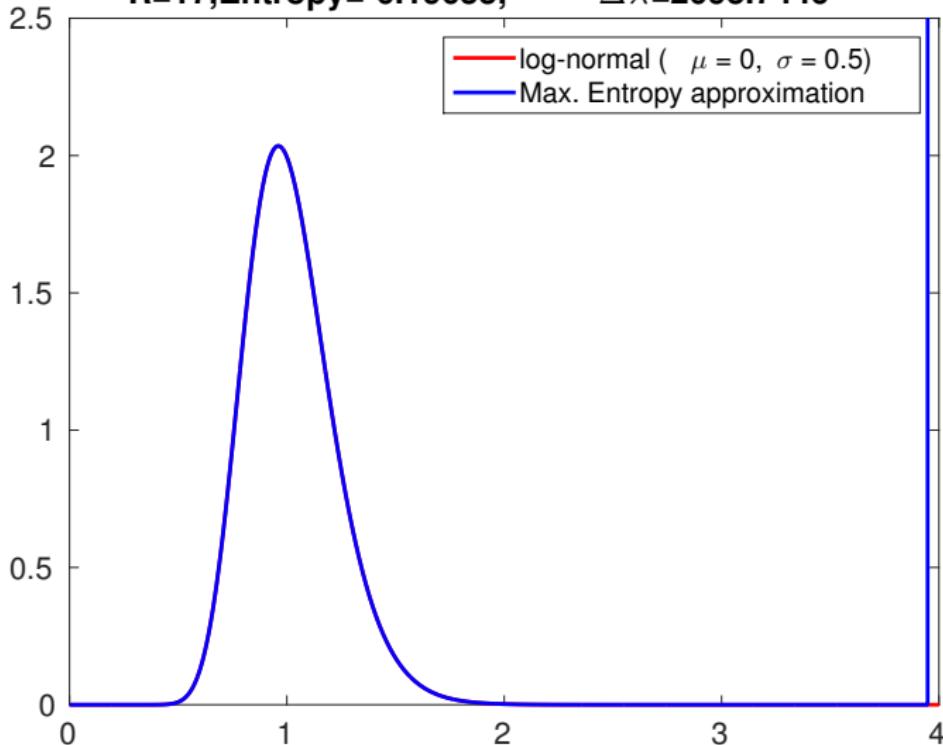


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=17$ , Entropy = -0.19055,

$\Delta \lambda = 2095.7446$

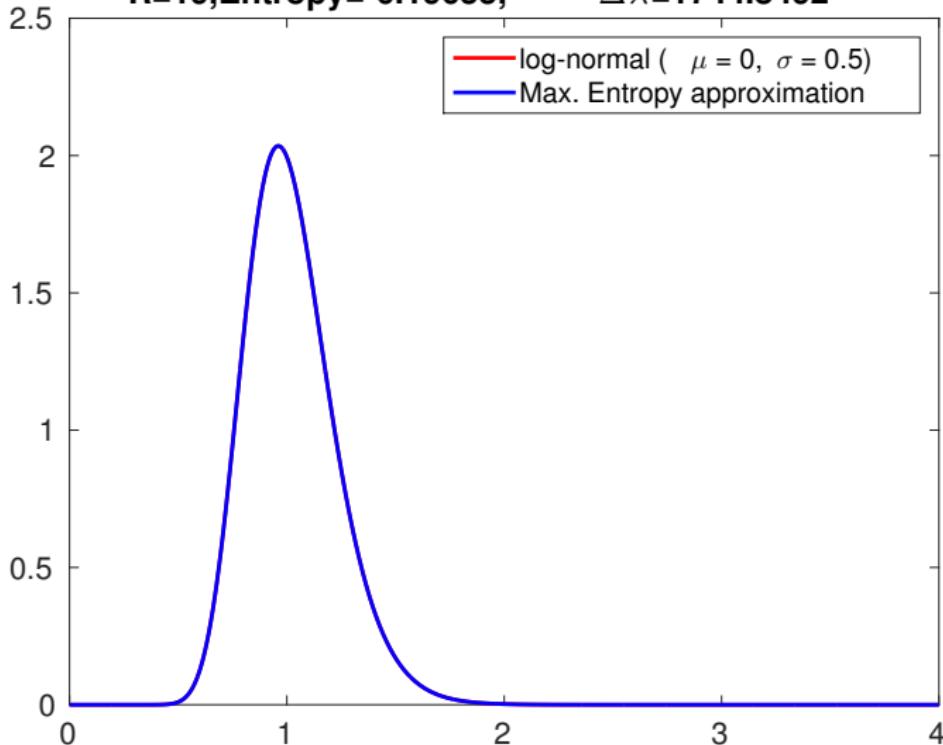


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

$R=19$ , Entropy = -0.19055,

$\Delta \lambda = 1744.8492$

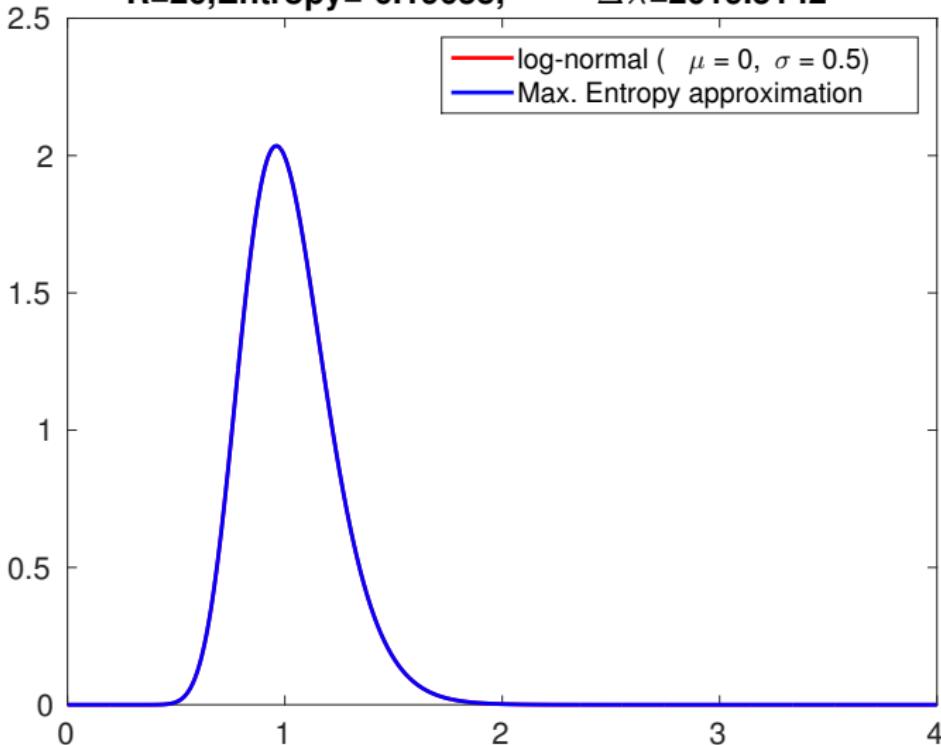


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!

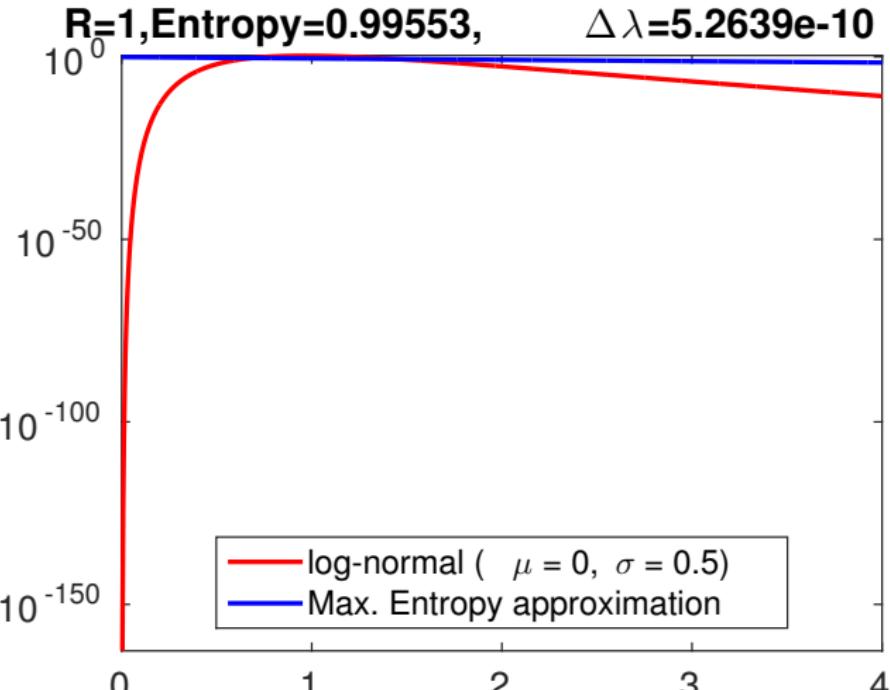
$R=20, \text{Entropy}=-0.19055,$

$\Delta\lambda=2019.5142$



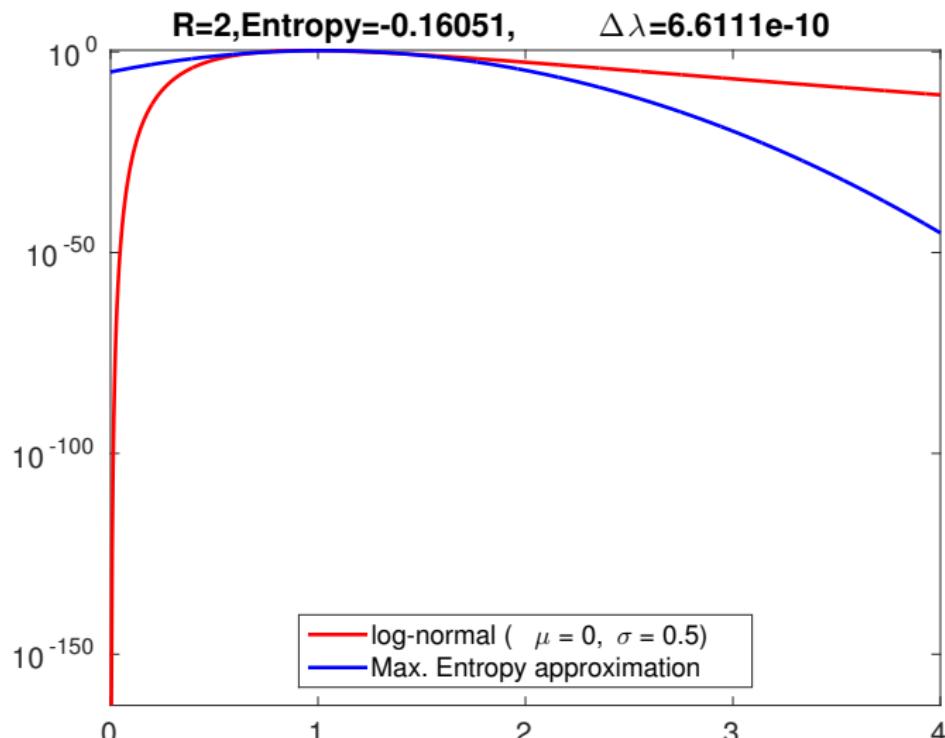
**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ ):**

- Still quite stable even without convergence!
- Entropy is still monotonously decreasing!



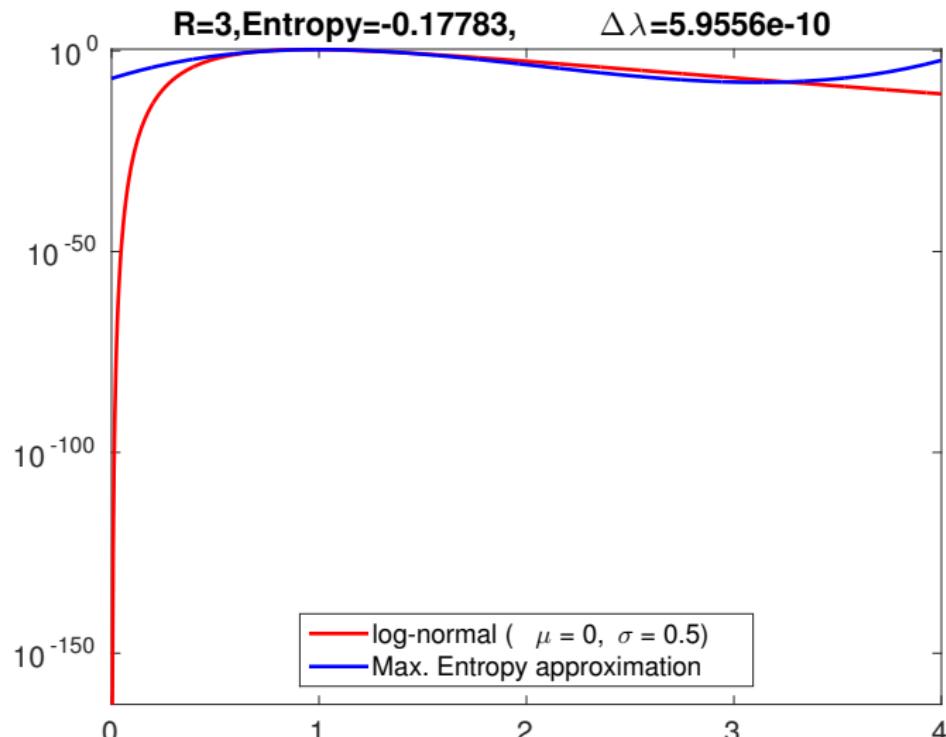
Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ , semilog):

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density



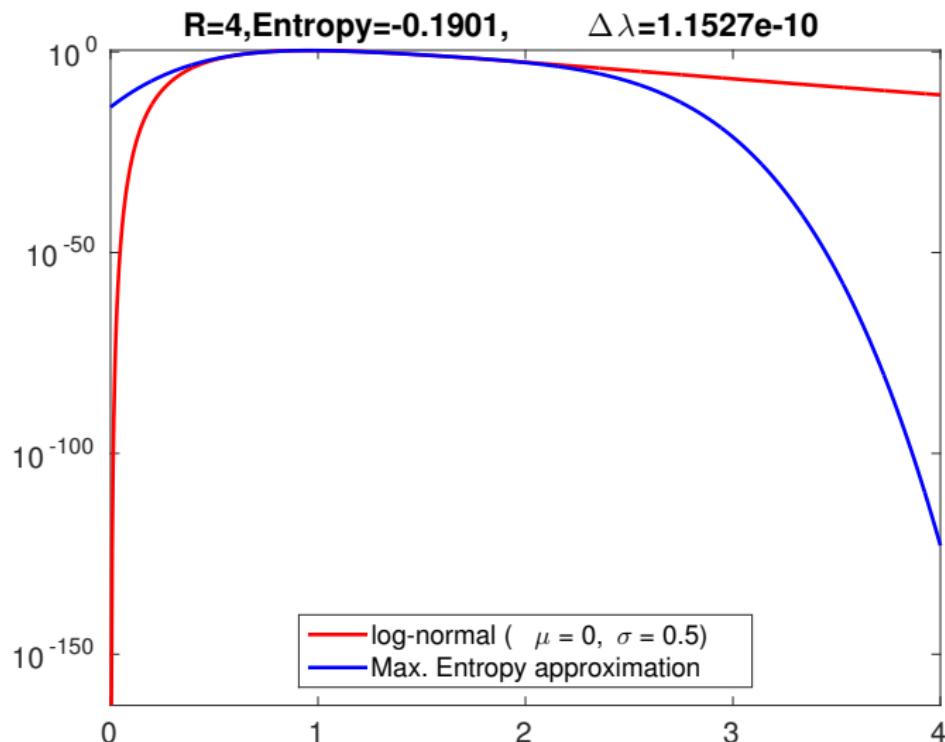
**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ , semilog):**

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density



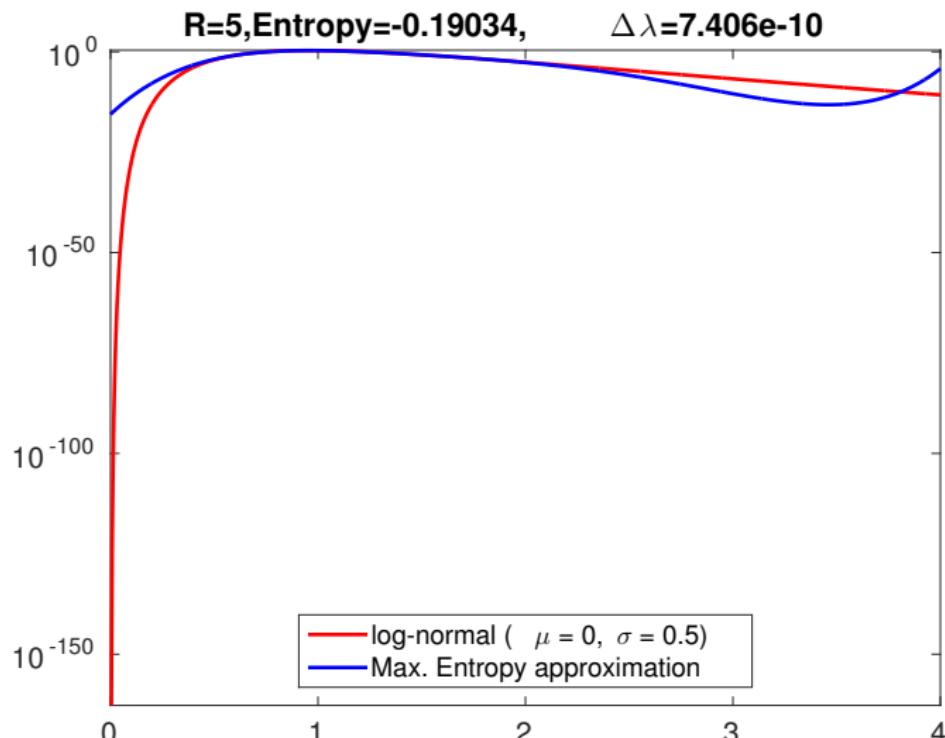
**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ , semilog):**

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density



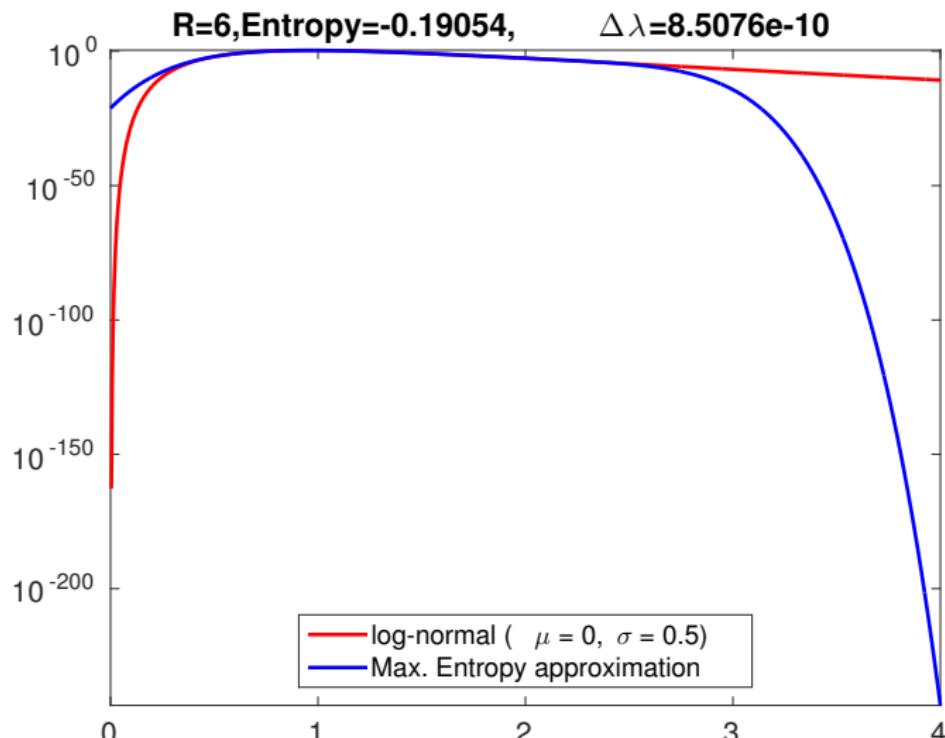
**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ , semilog):**

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density



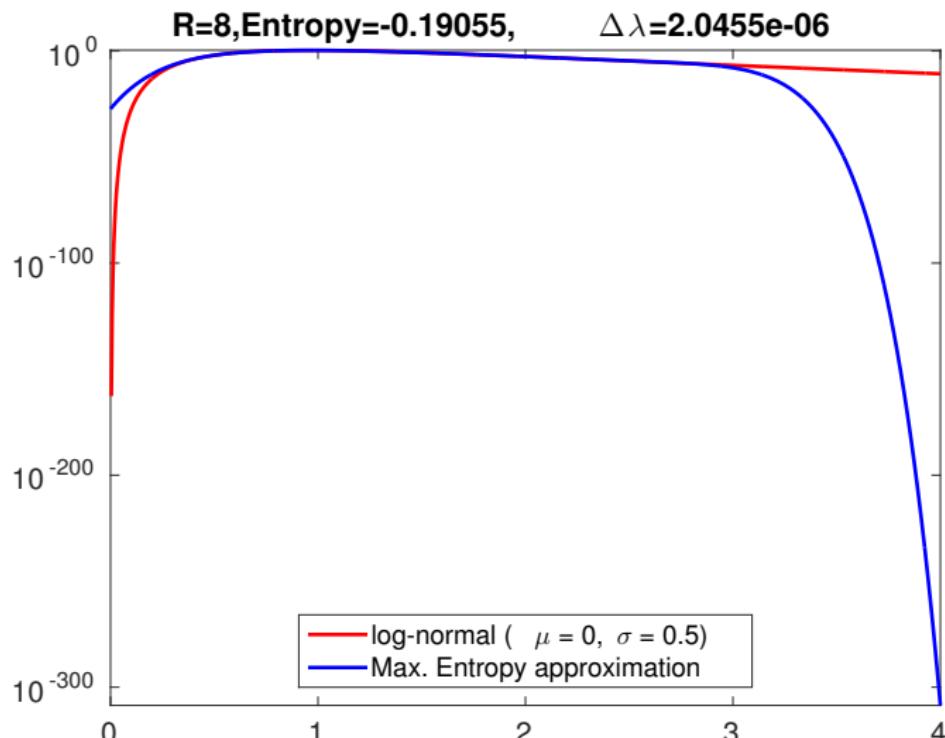
**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ , semilog):**

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density



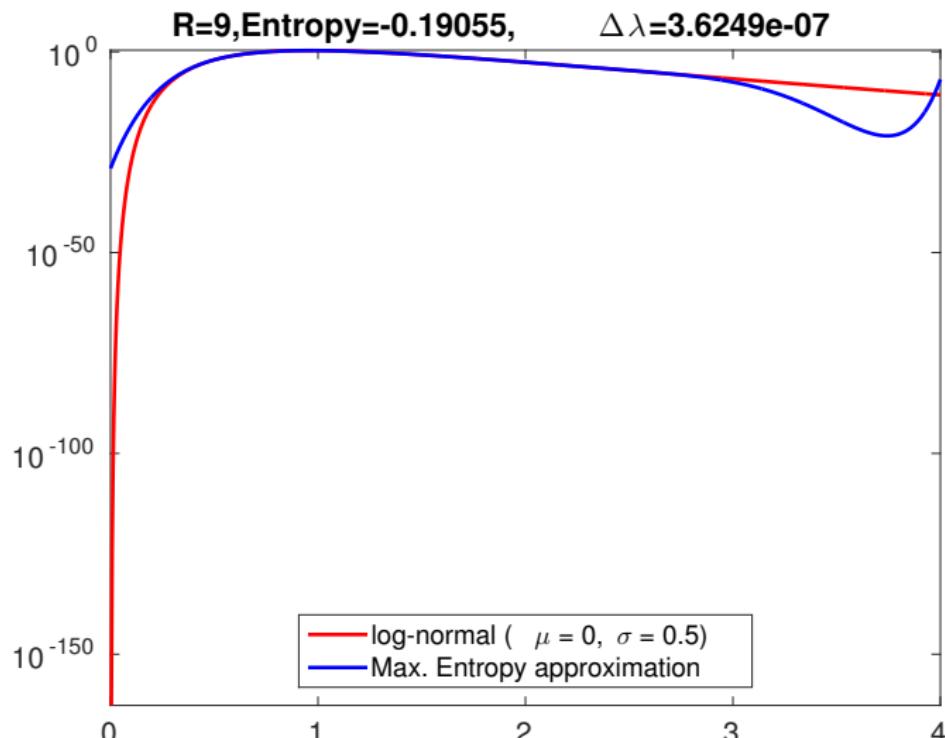
**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0, 4]$ , semilog):**

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density



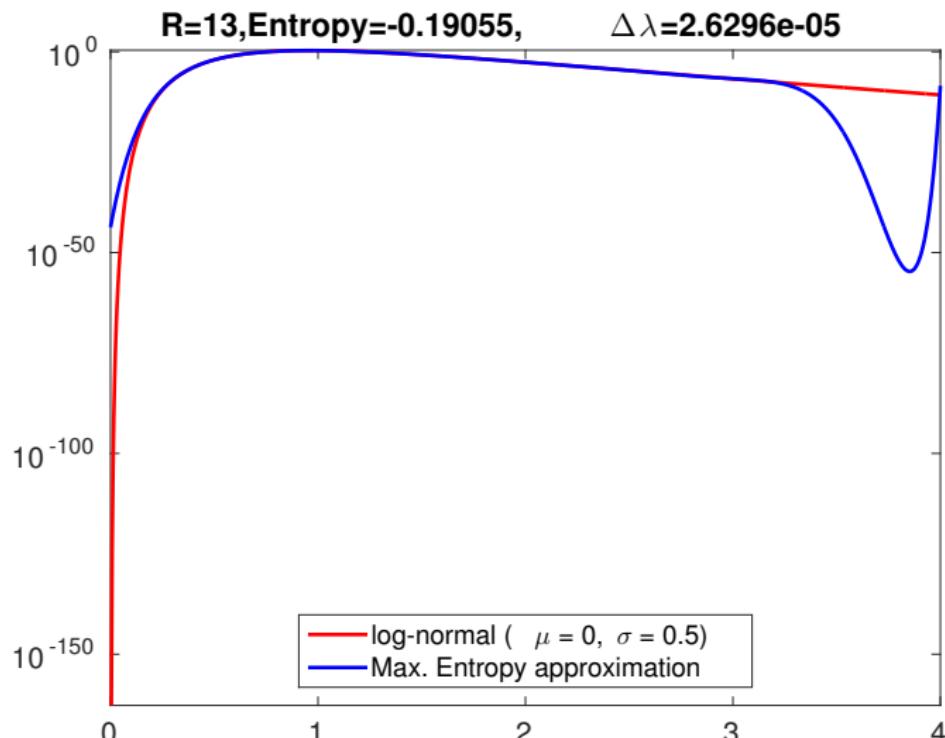
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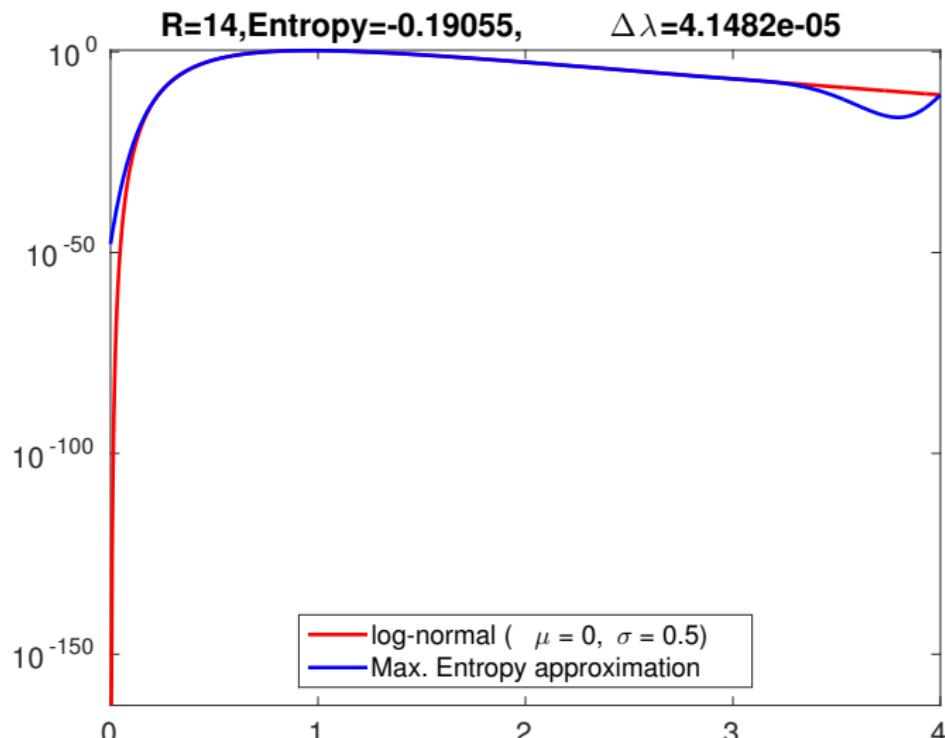
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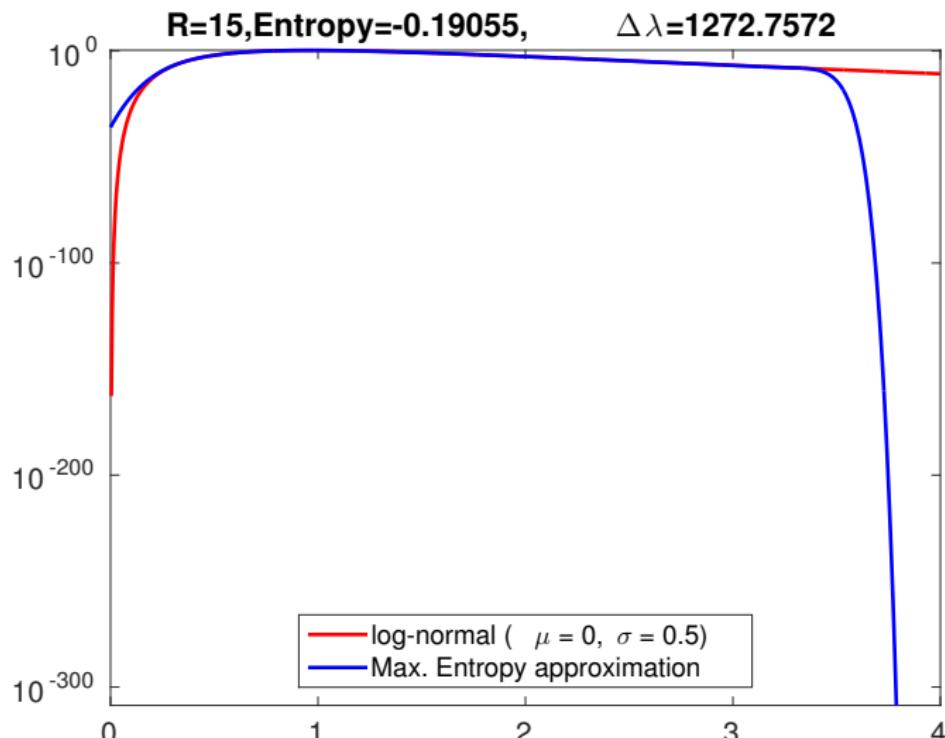
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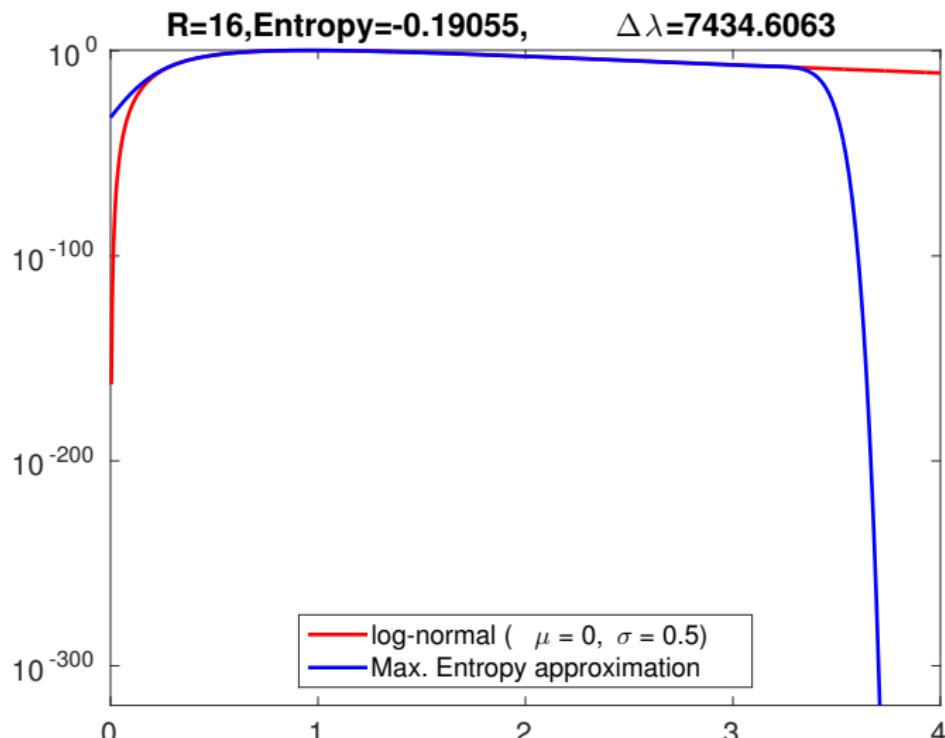
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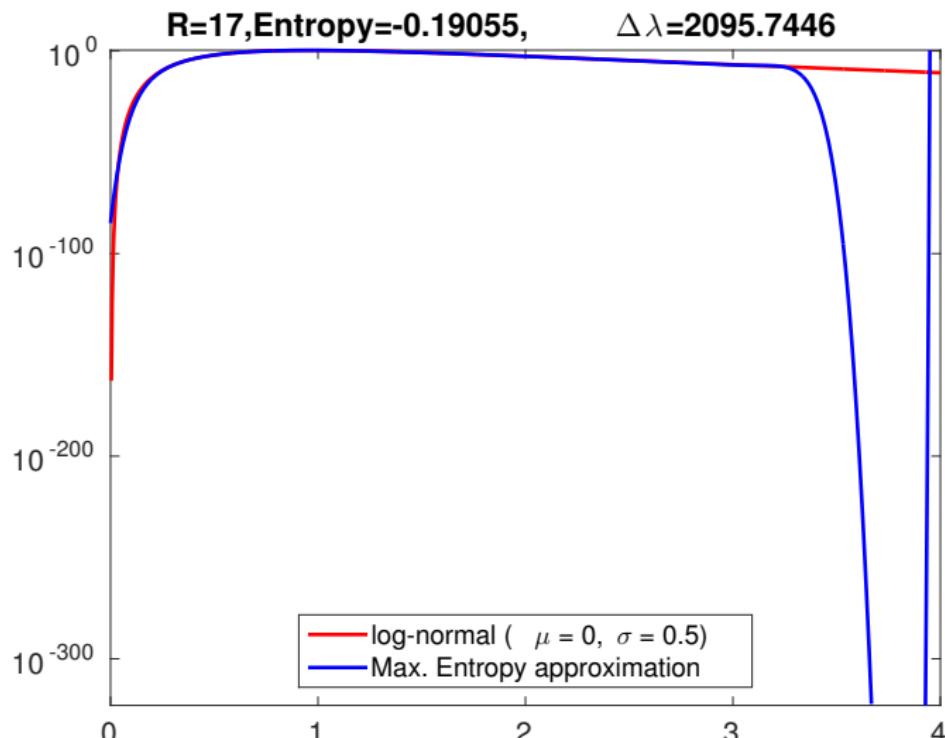
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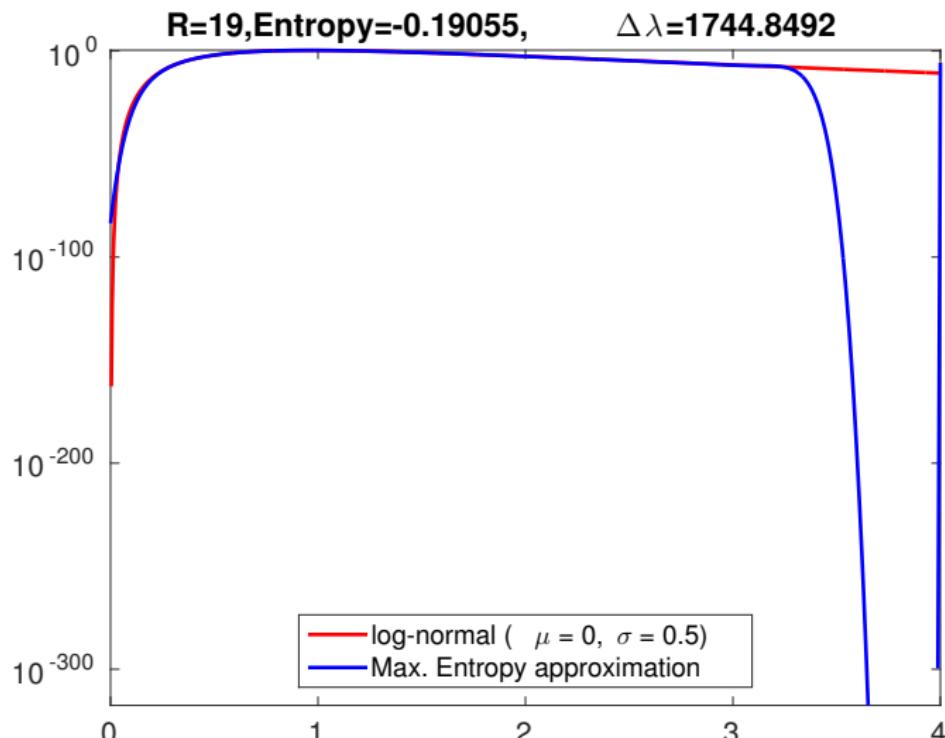
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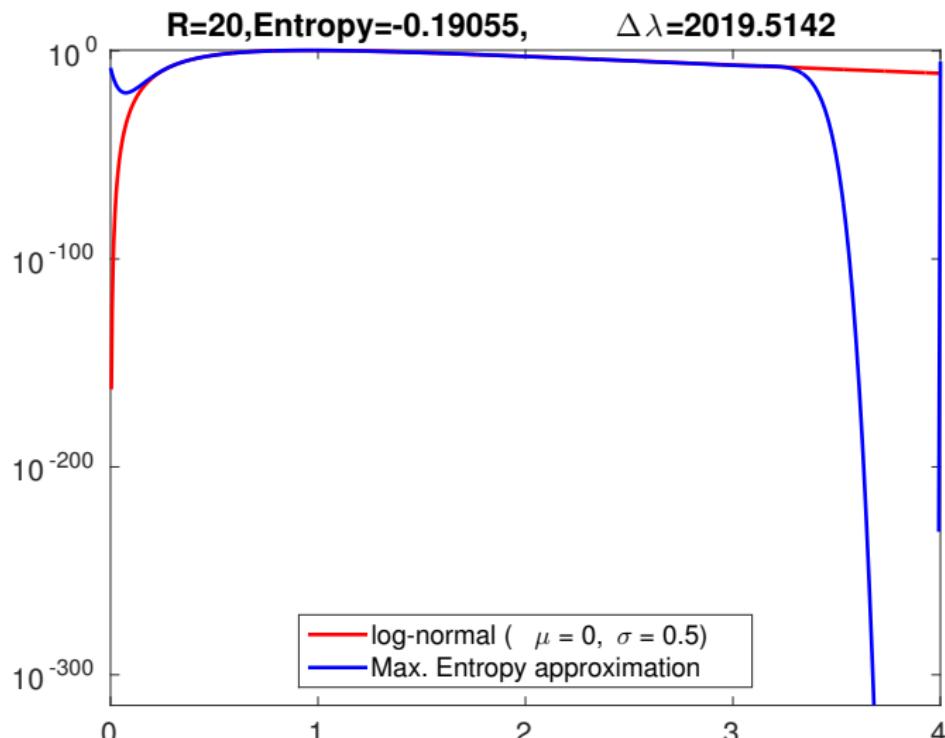
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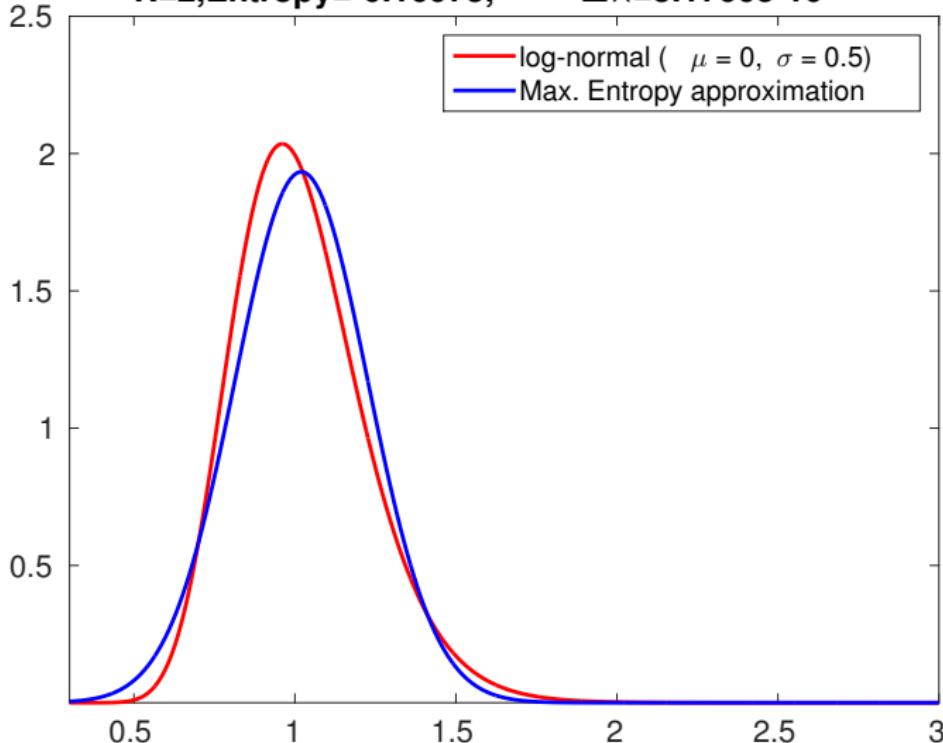


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0, 4]$ , semilog):**

- Oscillations in the negative domain
- $\Rightarrow$  stability of the density

$R=2, \text{Entropy}=-0.16075,$

$\Delta\lambda=5.1766e-10$

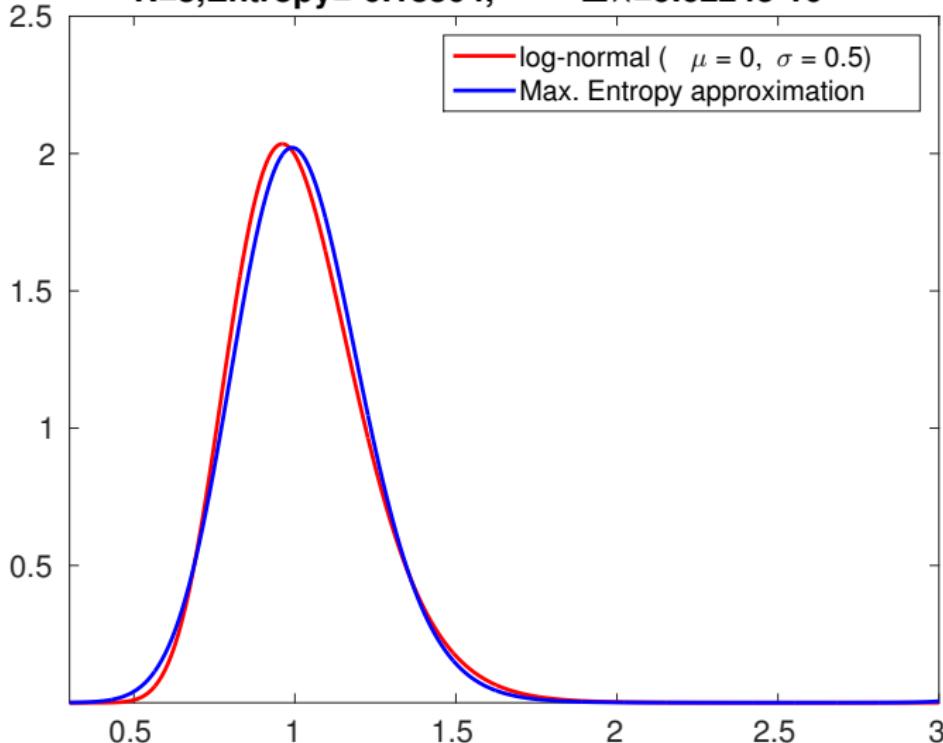


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=3, \text{Entropy}=-0.18304,$

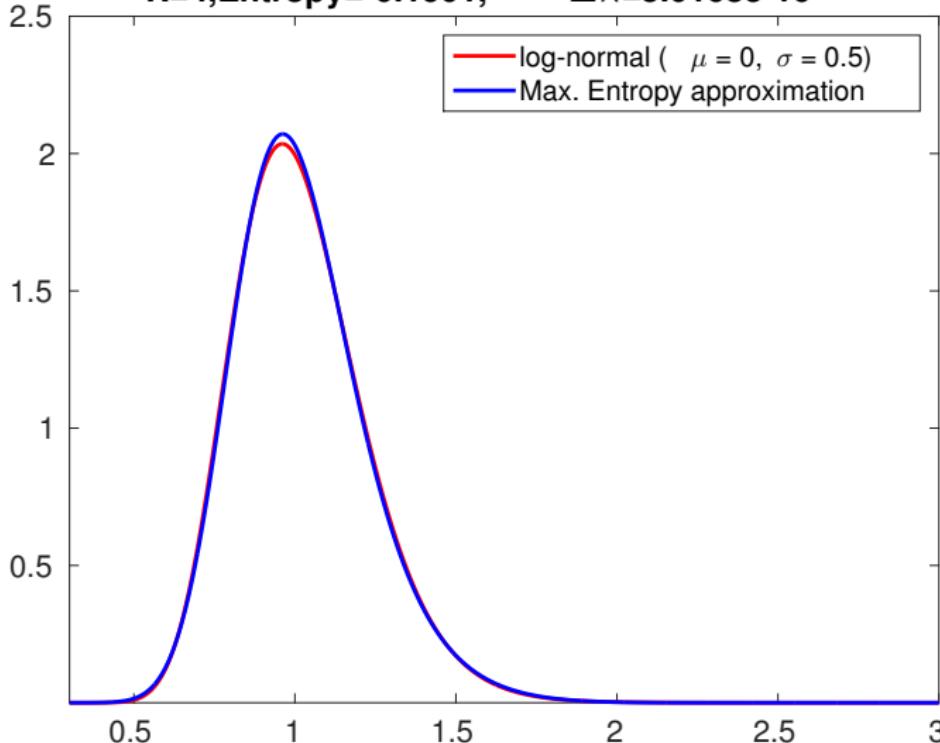
$\Delta\lambda=9.6224\text{e-}10$



**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

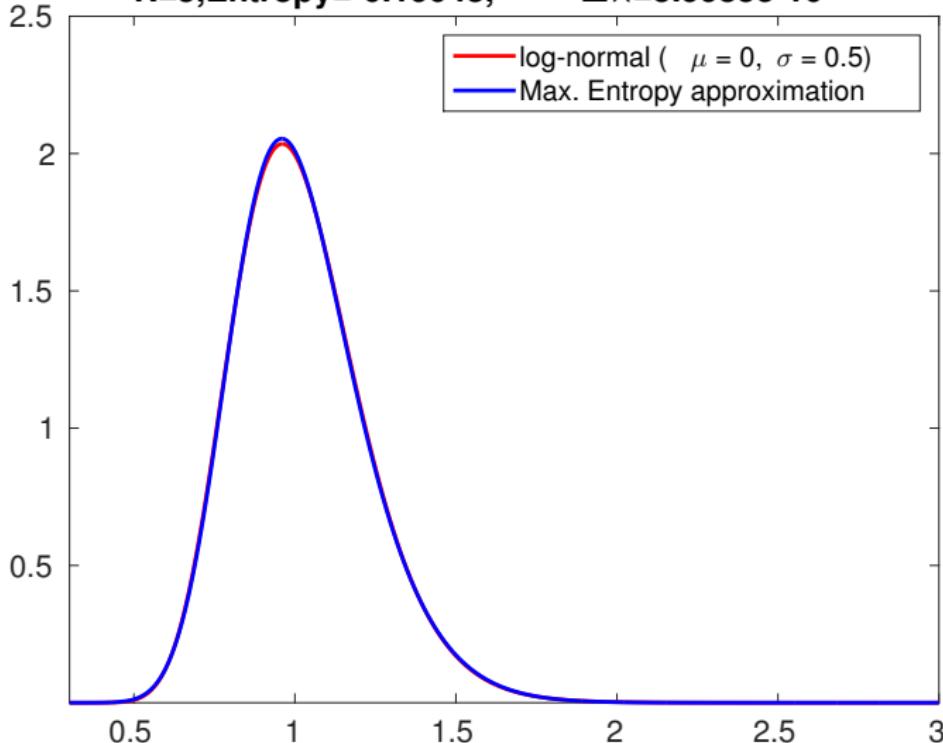
$R=4$ , Entropy = -0.1901,  $\Delta\lambda = 5.0168e-10$



**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=5, \text{Entropy}=-0.19043,$        $\Delta\lambda=5.9985e-10$

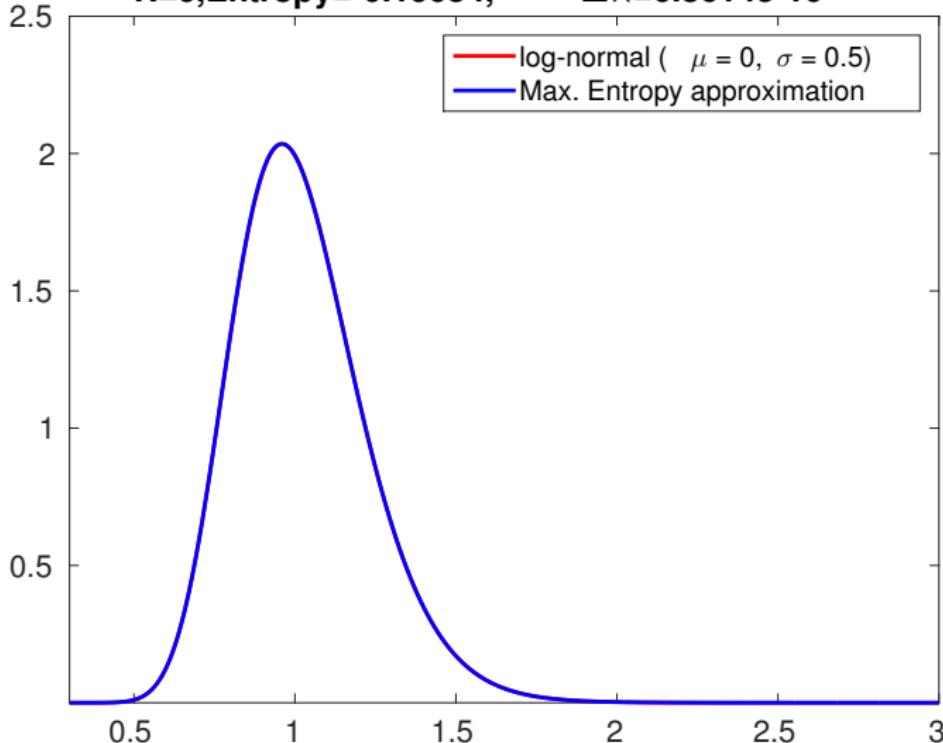


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=6, \text{Entropy}=-0.19054,$

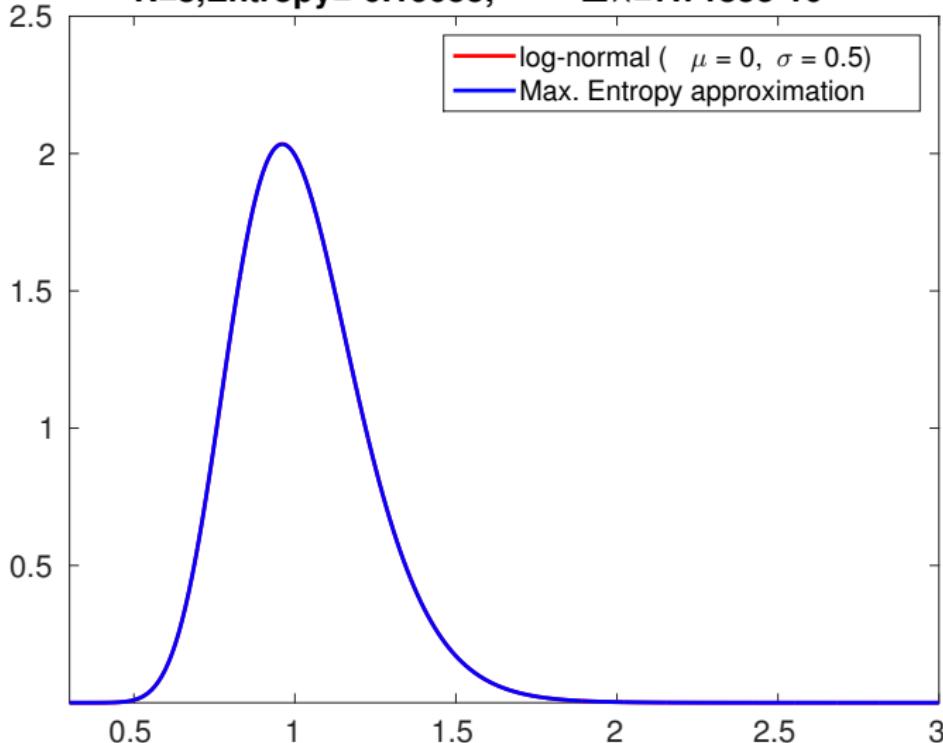
$\Delta\lambda=6.8614e-10$



**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=8, \text{Entropy}=-0.19055, \Delta\lambda=7.7185e-10$

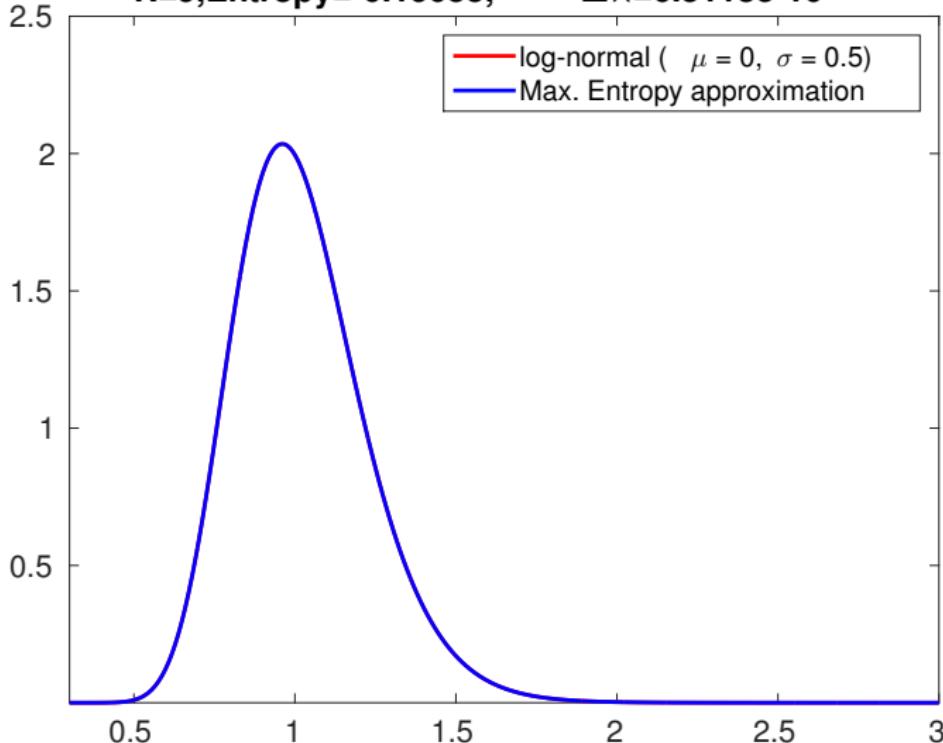


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=9, \text{Entropy}=-0.19055,$

$\Delta\lambda=6.5113e-10$

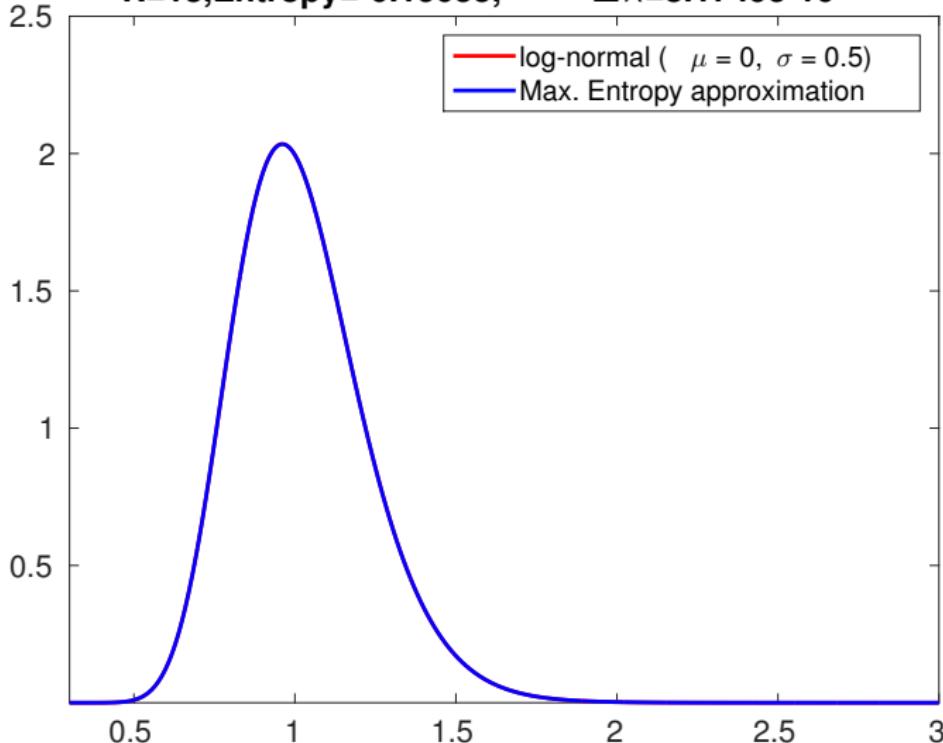


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=13, \text{Entropy}=-0.19055,$

$\Delta\lambda=8.1749e-10$

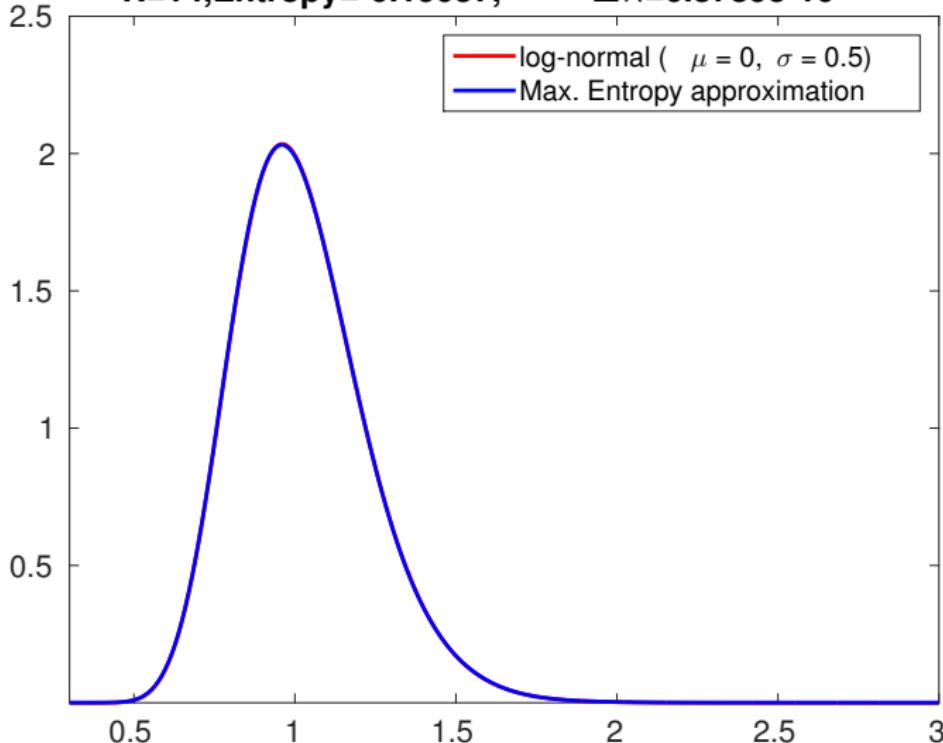


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=14$ , Entropy = -0.19057,

$\Delta \lambda = 6.3789 \times 10^{-10}$

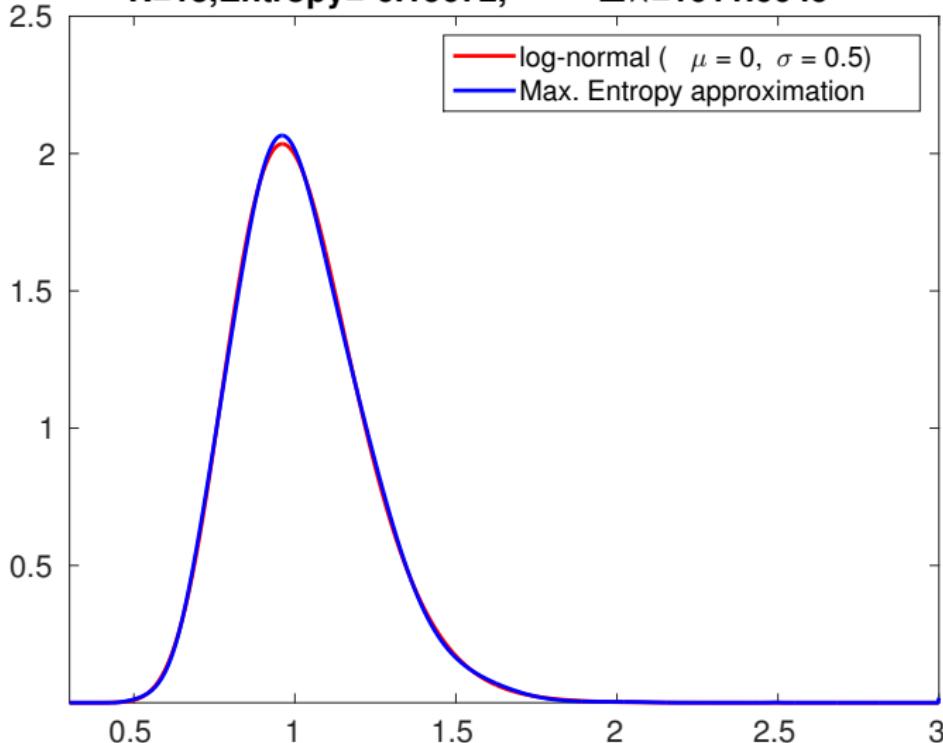


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=15, \text{Entropy}=-0.19072,$

$\Delta\lambda=1011.6049$

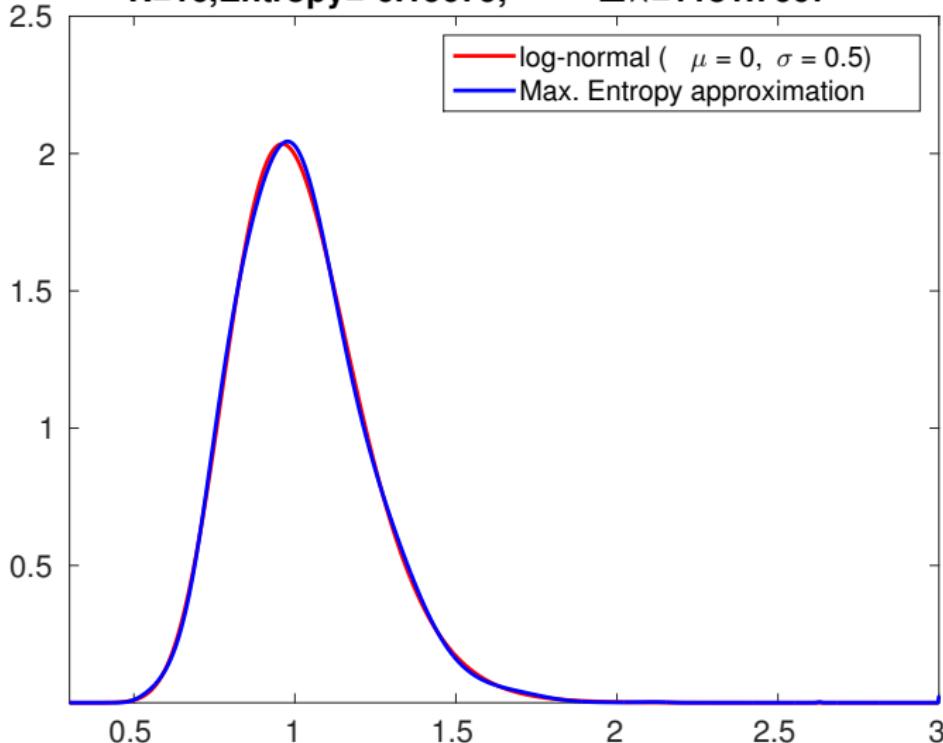


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=16, \text{Entropy}=-0.19079,$

$\Delta\lambda=1131.7607$

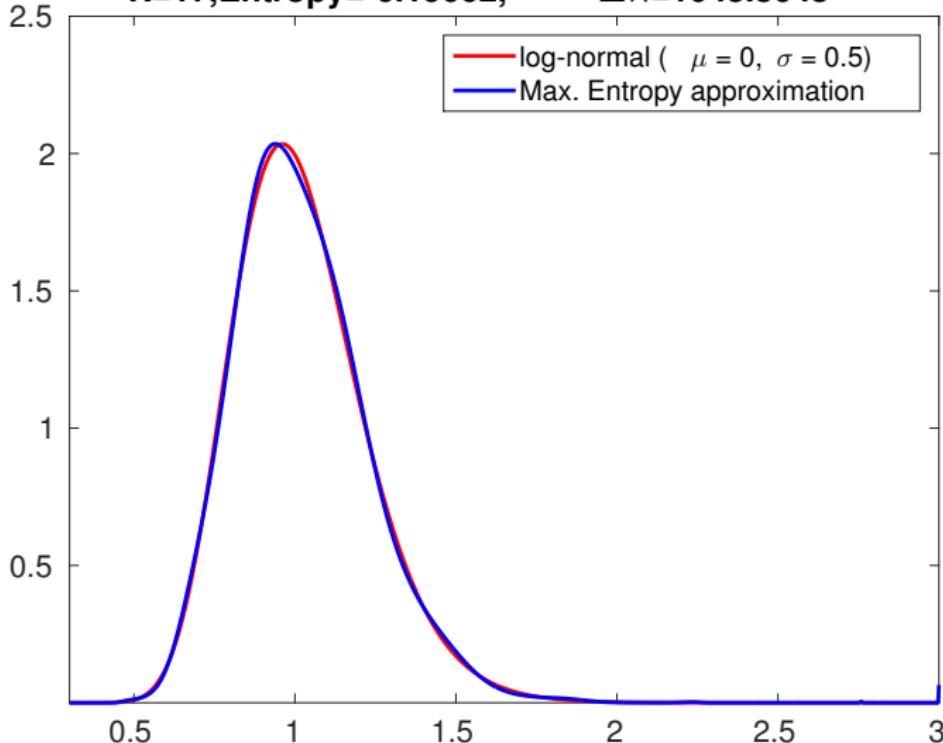


**Legendre Moments ( $\sigma = 0.2, [a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=17$ , Entropy = -0.19062,

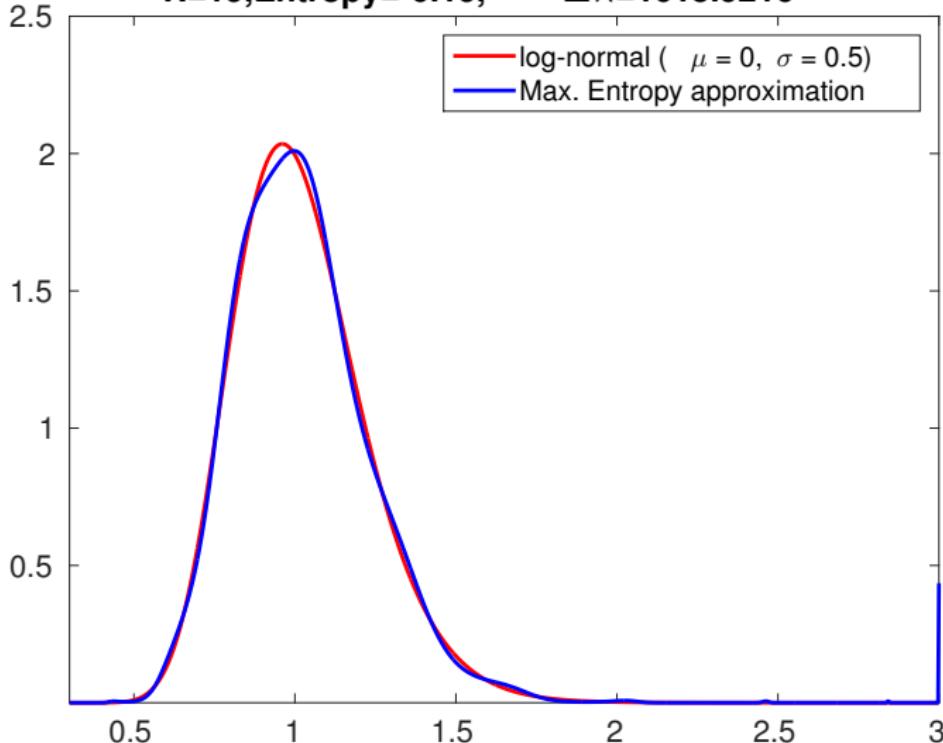
$\Delta \lambda = 1045.8643$



**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=19$ , Entropy=-0.19,  $\Delta \lambda = 1015.9216$

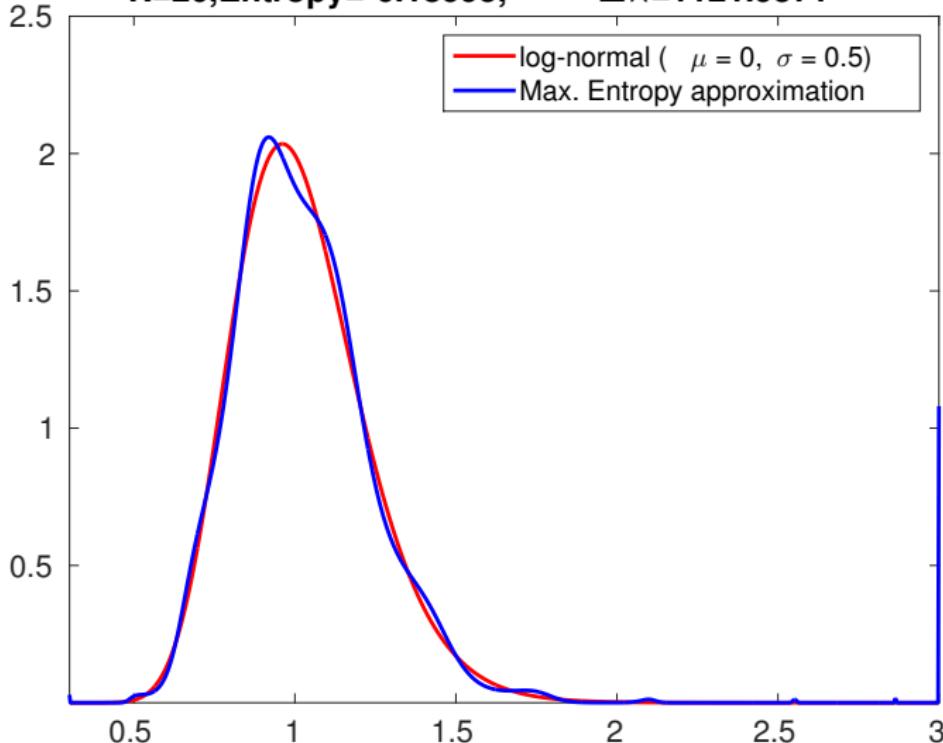


**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

$R=20, \text{Entropy}=-0.18995,$

$\Delta\lambda=1121.9571$



**Legendre Moments ( $\sigma = 0.2$ ,  $[a, b] = [0.3, 3]$ ):**

- Is stable and convergent for a bigger range  $R \leq 14!$
- Entropy is monoton. decr. (stat. accuracy of MC is reached?)

1. C. BIERIG AND A. CHERNOV, Convergence analysis of multilevel Monte Carlo variance estimators and application for random obstacle problems. *Numer. Math.* 130 (2015), no. 4, 579–613
2. C. BIERIG AND A. CHERNOV, Estimation of arbitrary order central statistical moments by the multilevel Monte Carlo Method. *Stoch. Partial Differ. Equ. Anal. Comput.* 4 (2016), no. 1, 3–40
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Many open questions...

Thank you for your attention!

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