# Ground-state clusters of two, three and four-dimensional $\pm J$ Ising spin glasses

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## Abstract

A huge number of independent true ground-state configurations is calculated for two-, three- and four-dimensional $\pm J$ spin-glass models. Using the genetic cluster-exact approximation method, system sizes up to $N = 20^2, 8^3, 6^4$ spins are treated. A "ballistic-search" algorithm is applied which allows even for large system sizes to identify clusters of ground states which are connected by chains of zero-energy flips of spins. The number of clusters $n_C$ diverges with $N$ going to infinity. For all dimensions considered here, an exponential increase of $n_C$ appears to be more likely than a growth with a power of $N$. The number of different ground states is found to grow clearly exponentially with $N$. A zero-temperature entropy per spin of $s_0 = 0.078(5)k_B$ (2d), $s_0 = 0.051(3)k_B$ (3d) respectively $s_0 = 0.027(5)k_B$ (4d) is obtained.

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## 1 *Introduction*

Spin-glass models [1] with discrete distributions of the interactions are believed to exhibit a rich ground-state ($T = 0$) landscape. So far only for very small systems of $N = 4^3$ spins the complete landscape has been analyzed [2]. In this work a new "ballistic search" algorithm is presented, which allows the treatment of much larger systems. As an application, two-, three-, and four-dimensional Edwards-Anderson (EA) $\pm J$ spin glasses are investigated. They consist of $N$ spins $\sigma_i = \pm 1$, described by the Hamiltonian

$$H \equiv -\sum_{\langle i,j\rangle} J_{ij}\sigma_i\sigma_j \quad . \tag{1}$$

The sum runs over all pairs of nearest neighbors. The spins are placed on $d = 2, 3, 4$-dimensional simple (square/ cubic/ hypercubic) lattices of linear size $L$ with periodic boundary conditions in all directions. Systems with quenched disorder of the interactions (bonds) are considered. Their possible values are $J_{ij} = \pm 1$ with equal probability. To reduce the fluctuations, a constraint is imposed, such that $\sum_{\langle i,j\rangle} J_{ij} = 0$. Since the Hamiltonian exhibits no external field, reversing all spins of a *configuration* (also called *state*) $\{\sigma_i\}$ results in a state with the same energy, called the *inverse* of $\{\sigma_i\}$. In the following, a spin configuration and its inverse are regarded as one single state.

Since the ground-state problem belongs to the class of NP-hard tasks [3], only algorithms with exponentially increasing running time are available. Currently it is possible to obtain a finite number of true ground states per realization up to $L = 56$ (2d),

$L = 14$ (3d) or $L = 8$ (4d) using a special optimization algorithm [4]. For the $\pm J$ model the number of existing ground states $n_{GS}$ per realization, called the *ground-state degeneracy*, grows exponentially with $N$. The reason is, that there are usually *free* spins, i.e. spins which can be flipped without changing the energy of the system. A state with $f$ independent free spins allows at least for $2^f$ different configurations of the same energy. Currently, it seems to be impossible to obtain all ground states for system sizes larger than $L = 5$. To overcome this problem in this work all *clusters* of ground states are calculated. A cluster is defined in the following way: Two ground state configurations are called *neighbors* if they differ only by the orientation of one free spin. All ground states which are accessible through this neighbor relation are defined to be in the same cluster. This means, one can travel through the ground states of one cluster by flipping only free spins. With the method presented here, the *ballistic search* (BS), it is not only possible to analyze large ground-state clusters, furthermore it allows to obtain the cluster landscape when having only a small subset of all ground states available. Additionally one can estimate the size of the clusters from this small number of sample states, as shown later on.

The number of clusters as a function of system size is also of interest on its own: for the infinitely-ranged Sherrington-Kirkpatrik (SK) Ising spin glass a complex configuration-space structure was found using the replica-symmetry-breaking mean-field (MF) scheme by Parisi [5]. If the MF scheme is valid for finite-dimensional spin glasses as well, then the number of ground-state clusters must diverge with increasing system size. On the other hand the droplet-scaling picture [6, 7, 8, 9] predicts that basically one ground-state cluster (and its inverse) dominates the spin-glass behavior. To address this issue a cluster-analysis was performed for small systems of one size $L = 4$ in three dimensions [2]. But an analysis of the size dependence of the number of clusters or even an investigation of two-/four-dimensional spin glasses has not been carried out before.

By the way, with the method presented here it is possible to calculate the entropy $S_0 \equiv \langle \ln n_{GS} \rangle k_B$ even for systems exhibiting a huge $T = 0$ degeneracy. The symbol $\langle \ldots \rangle$ denotes the average over different realizations of the bonds. Since the number of free spins is extensive, $s_0 \equiv S_0/N > 0$ holds for the $\pm J$ spin glass. For three-dimensional spin glasses, in [10] the ground-state entropy was estimated by computing exact free energies for systems of size $4 \times 4 \times M$ ($4 \leq M \leq 10$). In [11] a Monte-Carlo simulation and in [12, 13] multicanonical simulations were used to calculate $s_0$. Results for the ground-state entropy of two dimensional system were obtained with similar methods: numerically exact calculations of finite systems [14, 15], Monte-Carlo simulations[11, 16] and analytics methods [17, 18, 19]. For $d = 4$ the author is not aware of results for the ground-state entropy.

The paper is organized as follows: First the procedures used in this work are presented. Then the results for the number of clusters and the number of ground states as function of $N$ in two, three and four dimensions are shown. The last section a summarizes the results.

## 2 *Algorithms*

At first the optimization method applied here is stated. Then, for illustrating the problem, a simple method for constructing clusters of ground states is explained. In the main part the BS method for identifying clusters in systems exhibiting a huge degeneracy is presented and it is explained how this technique can be applied to estimate the size of these clusters.

The basic method used here for the calculation of spin-glass ground states is the cluster-exact approximation (CEA) algorithm [20], which is a discrete optimization method designed especially for spin glasses. In combination with a genetic algorithm [21, 22] this method is able to calculate true ground states [4]. Using this technique one does not encounter ergodicity problems or critical slowing down like in algorithms which are based on Monte-Carlo methods. Genetic CEA was already utilized to examine the ground states of two-, three- and four dimensional $\pm J$ spin glasses by calculating a small number of ground states per realization [4], while in this work the emphasize is on the study of the cluster landscape. Therefore, many ground states per random sample have to be obtained. Since the algorithm calculates only one independent ground state per run, a much larger computation effort was necessary.

Once many ground states are calculated the straight-forward method to obtain the cluster landscape works the following way: The construction starts with one arbitrary ground state. All its neighbors are added to the cluster. These neighbors are treated recursively in the same way: All their neighbors which are yet not included in the cluster are added. After the construction of one cluster is completed the construction of the next one starts with a ground state, which has not been visited so far.

The construction of the clusters needs only linear computer-time as function of $n_{SG}$ ($O(n_{SG})$), similar to the Hoshen-Kopelman technique [23], because each ground state is visited only once. Unfortunately the detection of all neighbors, which has to be performed at the beginning, is of $O(n_{SG}^2)$ since all pairs of states have to be compared. Even worse, all existing ground states must have been calculated before. As e.g. a $5^3$ system may exhibit already more than $10^5$ ground states, this algorithm is not suitable.

The basic idea of the ballistic-search algorithm is to use a *test*, which tells whether two ground states are in the same cluster. The test works as follows: Given two independent replicas $\{\sigma_i^\alpha\}$ and $\{\sigma_i^\beta\}$ let $D$ be the set of spins, which are different in both states: $D \equiv \{i|\sigma_i^\alpha \neq \sigma_i^\beta\}$. Now BS tries to build a path of successive flips of free spins, which leads from $\{\sigma_i^\alpha\}$ to $\{\sigma_i^\beta\}$ while using only spins from $D$. In the simplest version iteratively a free spin is selected randomly from $D$, flipped and removed from $D$. This test does not guarantee to find a path between two ground states which belong to the same cluster, since it may depend on the order the spins are selected whether a path is found or not. It only finds a path with a certain probability which depends on the size of $D$. It turns out that the probability decreases monotonically with $|D|$. For example for $N = 8^3$ the method finds a path in 90% of all cases if the two states differ by 34 spins. More analysis can be found in [24].

The algorithm for the identification of clusters using BS works as follows: the basic idea is to let a ground state represent that part of a cluster which can be found using BS with a high probability by starting at this ground state. If a cluster is large it has to be represented by a collection of states, such that the whole cluster is "covered". For example a typical cluster of a $8^3$ spin glass consisting of $10^{16}$ ground states is usually represented by only some few ground states (e.g. two or three). A detailed analysis of how many representing ground states are needed as a function of cluster and system size can be found in [24]. The algorithm holds in memory a set of clusters consisting each of a set of representing configurations. At the beginning the cluster set is empty. Iteratively all available ground states $\{\sigma_i\}$ are treated: For all representing configurations the BS algorithm tries to find a path to the current ground state or to its inverse. If no path is found, a new cluster is created, which is represented by the actual configuration treated. If $\{\sigma_i\}$ is found to be in exactly one cluster nothing special happens. If $\{\sigma_i\}$ is found to be in more than one cluster all these clusters are merged into one single cluster, which is now represented by the union of the states which have represented all clusters affected by the merge.

The BS identification algorithm has some advantages in comparison with the straight-forward method: since each ground-state configuration represents many ground states, the method does not need to compare all pairs of states. Each state is compared only to a few number of representing configurations. Thus, the computer time needed for the calculation grows only a little bit faster than $O(n_{SG}n_C)$ [24], where $n_C$ is the number of clusters which is much smaller than $n_{SG}$. Consequently, large sets of ground states, which appear already for small system sizes like $L = 5$, can be treated. Furthermore, the ground-state cluster landscape of even larger systems can be analyzed, since it is sufficient to calculate a small number of ground states per cluster. One has to ensure that really all clusters are found, which is simply done by calculating enough states, but this is still only a tiny fraction of all ground states [24]. Also one has to be sure that all clusters are identified correctly. This is not guaranteed immediately, since for two ground states belonging to the same cluster there is just a certain probability that a path of free flipping spins connecting them is found. But this poses no problem, because once at least one state of a cluster has been found, many more states can be obtained easily by just performing a $T = 0$ Monte-Carlo simulation starting with the initial state. By increasing the number of states available more and more, the probability that all clusters have been identified correctly approaches very quickly one. Detailed tests can be found in [24]. For all results presented here, the number of available ground states has been increased so far, such that each cluster has be identified correctly with a probability of more than 0.99.

Once all ground states are grouped into clusters, their sizes have to be obtained to calculate the total number of states and the entropy. If only some ground states per cluster are available, the size cannot be evaluated by simply counting the states. Then a variant of BS is used to perform this task. Given a state $\{\sigma_i\}$, free spins are flipped iteratively, but each spin not more than once. During the iteration additional free spins may be generated or destroyed. When there are no more free spins left, the process stops. One counts the number of spins that has been flipped. By averaging over several tries and several ground states of a cluster one obtains an average value, denoted with $l_{\max}$. It can be shown that this quantity represents the size $n_C$ of a cluster very well and is more accurate than simpler measures such as the average number of static free spins. By analyzing all ground states of small systems a $n_C = 2^{\alpha l_{\max}}$ behavior is found, with $\alpha \in [0.85, 0.93]$ depending on the dimension of the system. These results will be exposed in the next section. A similar method for estimating the cluster sizes is presented in [25]. There three heuristic fitting-parameters are needed, but they are universal for all system dimensions.

# 3 *Results*

First, the results for three-dimensional systems are given. In the second and third part two- and four-dimensional spin glasses are investigated.

In 3d, for system sizes $L = 3, 4, 5, 6, 8$ large numbers of independent ground states were calculated using genetic CEA. Usually 1000 different realizations of the disorder were considered. Tab. I shows the number of realizations $n_R$ and the number of independent runs $r$ per realization for the different system sizes $L$. For the small systems sizes (and for 100 realizations of $L = 5$) many runs plus an additional local search were performed to calculate *all* ground states. For the larger sizes $L = 5, 6, 8$ the number of ground states is too large, so it is only possible to try to calculate at least one ground state per cluster. It is highly probable that all clusters were detected, except for $L = 8$, where for about 25% of the realizations some small cluster may have been missed [24]. This problem is not related to the design of the ballistics search method. It is due to the enormous computational effort needed for generating the ground states of the largest systems, so only a restricted number of runs can be performed. Since the probability that a certain cluster is found in a run of the genetic CEA algorithm decreases with the size of the cluster [26], ground states belonging to small clusters occur only rarely. Even by doubling the number of runs for $L = 8$, this fraction is estimated to fall only to 20 %.

The ground states were grouped into clusters using the ballistic-search algorithm. The number of states per cluster was sufficiently large, so that only with a probability of less than $10^{-3}$ some configurations from a large cluster may be mistaken for belonging to different clusters [24]. The average number $n_C$ of clusters is shown in the fourth column of Tab I. In Fig. 1 the result is shown as a function of the number $N$ of spins. By visualizing the results using a double-logarithmic plot (see inset) one realizes that $n_C$ seems to grow faster than any power of $N$. The larger slope in the linear-logarithmic plot for small systems may be a finite-size effect. Additionally, for $L = 8$ there is a large probability that some small clusters are missed, explaining the smaller slope there. Summarizing, our data favor an exponential increase of $n_C(N)$.

To calculate the ground-state entropy, the size of the clusters have to be known. For the small systems, this can be done just by counting. For larger system sizes it is not possible to obtain all states, so the method using the dynamical number $l_{\max}$ of free spins is applied, as explained before. In Fig. 2 the cluster size for small systems is shown as a function of $l_{\max}$ with a logarithmically scaled y-axis. A $n_C = 2^{\alpha l_{\max}}$ dependence is visible very well, yielding $a = 0.90(5)$.

By summing up all cluster sizes for each realization the ground-state degeneracy $n_{GS}$ is obtained. Its average is shown in the fifth column of the table. The quantity is plotted in Fig. 3 as a function of $N$. The exponential growth is obvious.

The result for the average ground-state entropy per spin is shown in the last column of Tab. I. The number for $L = 4$ is within two standard deviations of $s_0 = 0.073(7)k_B$ which was found in [2], where 200 realization were treated. By fitting a function of the form $s_0(L) = s_0(\infty) + a * L^{-\beta}$ a value of $s_0(\infty) = 0.0505(6)k_B$ is obtained. In [10] $s_0 = 0.04(1)k_B$ was estimated for systems with periodic boundary conditions only in two directions, which may be the reason for the smaller result. The value found by a Monte-Carlo simulation $s_0 = 0.062k_B$ [11] for systems of size $20^3$ is much larger. The deviation is presumably caused by the fact that it was not possible to obtain true ground states for systems of that size, i.e. too many states were found. The results from multicanonical simulations $s_0 = 0.046(2)k_B$ [12] and $s_0 = 0.0441(5)k_B$ [13] are a little bit lower than the results obtained here. This may indicate that not all ground states are found using that simulation procedure.

The result for the entropy does not suffer from the fact, that some ground-state clusters may have been missed for $L = 8$: the probability for finding a cluster using genetic CEA grows with the size of the cluster [26]. This implies that the clusters, which may have been missed, are considerably small, so the influence on the result is negligible. The largest source of uncertainty is caused by the assumption, that the size of a cluster grows like $2^{\alpha l_{\max}}$. The error of the constant $\alpha$ enters linearly the result of the entropy. To estimate the influence of this approximation, for the three smallest systems sizes, where the entropy was obtained exactly, $s_0$ was calculated using estimated cluster sizes as well. For all three cases the result was equal to the exact values within error bars. The final result quoted here is $s_0 = 0.051(1)$.

Now we concentrate on two-dimensional systems. For system sizes $L = 5, 7, 10, 14, 20$ large numbers of independent ground states were calculated using genetic CEA, up to $10^4$ runs per realization were performed. Usually 1000 different realizations of the disorder were considered, except for $L = 20$, where only 96 realizations could be treated For the small systems sizes $L = 5, 7$, many runs plus an additional local search were performed to calculate *all* ground states. For the larger sizes $L = 10, 14, 20$ the number of ground states is too large, so we restrict ourselves to calculate at least one ground state per cluster. The probability that some clusters were missed is higher for two dimensions than for the $d = 3$ case, because the ground-state degeneracy grows faster with the system size: for small systems sizes $L \leq 10$ it is again highly probable that all clusters have been obtained. For $L = 14$ some small clusters may have been missed for about 30% of all realizations, while for $L = 20$ this fraction raises even to 60%. This is due to the enormous computational effort needed for the largest systems. For the $L = 20$ realizations a total computing time of more than 2 CPU-years was consumed on a cluster of Power-PC processors running with 80MHz.

The results for $d = 2$ are shown in Tab. II. The number of clusters $n_C$ as a function of system size is plotted in Fig. 4. Again it is more likely that $n_C$ grows exponential than an algebraic growth.

Similar to the $d = 3$ case, the cluster sizes $V$ can be obtained directly for small systems. For estimating $V$ in larger systems, again the $\alpha$ parameter has been obtained. The average size of a cluster as a function of $l_{max}$ is shown in Fig. 5 resulting in $\alpha = 0.85(5)$. With this parameter the ground-state degeneracy as a function of $N$ can be calculated, see Fig. 6. Similar to the $d = 3$ case, the exponential growth is obvious. The resulting entropy is shown in the inset. By a finite-size extrapolation to the infinite system, a value of $s_0 = 0.078(5)$ is obtained. In [14] $s_0 \approx 0.075k_B$ was estimated by using a recursive method to obtain numerically exact free energies up to $L = 18$. The result of $s_0 \approx 0.07k_B$ found in [15] is even slightly lower. The value found by a Monte-Carlo simulation $s_0 \approx 0.1k_B$ [11] for systems of size $80^2$ is much larger. The deviation is presumably caused by the fact that it was not possible to obtain true ground states for systems of that size, i.e. too many states were visited. Recent results are more accurate: by applying the replica Monte Carlo method [16] a value of $s_0 = 0.071(7)$ was obtained. A transfer matrix calculation [17] resulted in $s_0 = 0.0701(5)$. By using a Pfaffian method $s_0 = 0.0704(2)$ [18] respectively $s_0 = 0.0709(4)$ [19] was obtained. The most recent values are smaller than the entropy found in this work. The reason may be that larger systems could be treated (up to $L = 256$ in [18, 19]), while here an extrapolation has been performed with systems of size $L \leq 20$. At least, the value $s_0[L = 22] = 0.079(1)$ is comparable to the value of $s_0[L = 32] = 0.0780(8)$ found in [18]. Additionally, the fact that for the other works the number of antiferromagnetic bonds fluctuates from sample to sample while it is kept fixed here may have an influence as well. This was tested by calculating ground states for small systems ($L \leq 10$), where each bond has a probability 0.5 of being (anti-) ferromagnetic. In this case the entropy turned out to to 5-10 % below the values found above. For large system sizes, which are out of range for the method presented here, this effect should decrease.

In the last part we turn to four-dimensional $\pm J$ spin glasses. Because of the huge computational effort, $N = 6^4$ is the largest size which could be considered and a reasonable statistics could be only obtained for $L = 5$, since one $L = 6$ run takes several CPU weeks. For details, see Tab. III.

The number of clusters as a function of $N$ is displayed in Fig. 7. Here, even more clusters seem to have been missed than in the two-, and three-dimensional cases. But again, the data-basis is large enough that an exponential increase of the number of clusters seems possible.

The dependence of the cluster size on the number of flips of free spins could be studied only for the smallest system size. Even for $L = 4$ the number of ground states can grow beyond $10^6$, preventing a reliable analysis. From the $L = 3$ data (see Fig. 8) $a = 0.93(3)$ has been estimated.

In the final figure (Fig. 9) the resulting degeneracy is shown. Here, the small numbers of ground states, which could be calculated with reasonable effort, already have an influence on the results. For the largest size, the exponential growth of the number of ground states with system size is not visible. Please note that in general the average $n_{GS}$ is dominated by few samples having a large number of ground states. For $L = 6$, because of the small number of realizations, these realizations were not generated within 10 samples. This explains the deviation from the exponential growth.

For the entropy (see inset of Fig.9 ), rare samples have less influence since the logarithm of the number of states is averaged. Consequently, the value of $s_0 = 0.027(5)$, which again was obtained by a finite-size scaling fit, is much more reliable.

As we have seen, the $\alpha$-parameter increases with growing dimension. That means that the spins contributing to the ground-state degeneracy become more and more independent, the limit $\alpha = 1$ corresponds to the case were all free spins do not interact with each other. This can be understood from the decrease of the ground-state entropy. From $d = 2$ to $d = 4$ $s_0$ drops from 0.078 to 0.027. Thus, with growing dimension the number of spins contributing to the ground state degeneracy decreases quickly, so it becomes less likely that these spins are neighbors. This effect is stronger than the increase of the number of neighbors per spin from 4 in $d = 2$ to 8 in $d = 4$.

## 4 *Conclusion*

True ground states of two-, three- and four-dimensional $\pm J$ spin glasses have been calculated using genetic cluster-exact approximation. For each realization many independent ground states have been obtained, leading to an enormous computational effort: several months of running 32 PowerPC processors on a parallel computer were necessary. Clusters of ground states have been investigated, which are defined to be the sets of ground-state configurations which can be accessed from each other by flipping only free spins. The ballistic-search method has been presented, which allows the fast identification of very large clusters. It can be assured easily that the ground-state clusters found in this way have been identified correctly. It should be pointed out that this method is not a tool for the *calculation* of ground states of large systems, but it allows for a detailed *analysis* of highly degenerate ground-state landscapes. Indeed, it is possible to calculate clusters of systems when only a small fraction of their states is available. The method should be extendable to similar clustering-problems. A variant of the technique is used to estimate the size of clusters.

Ground-state clusters for systems of size up to $L = 20$ (2d), $L = 8$ (3d) and $L = 6$ (4d) have been calculated. It means that, in the case of three-dimensions, these realizations are ten times larger and have $10^{12}$ times more ground states than the systems treated in [2]. For the other dimensions similar studies even have not been performed before at all. The number of clusters and the degeneracy as a function of the number of spins $N$ were evaluated. It appears that both quantities are growing exponentially with $N$ for all three cases $d = 2, 3, 4$. Consequently, it seems unlikely that even larger systems can be treated accordingly in the near future. The ground-state entropy per spin was found to be $s_0 = 0.078(5)k_B$ (2d), $s_0 = 0.051(1)k_B$ (3d) respectively $s_0 = 0.027(5)k_B$ (4d). It should be stressed that the result for the entropy does not depend on the way a cluster is defined. The specific definition given here is only a tool, which allows the treatment of systems exhibiting a huge ground-state degeneracy. If ground states had colors, they could be grouped according their colors as well instead of performing a clustering according their neighbor relationship.

With the method presented here, its is only possible to study the top level clustering of the ground states. It is not possible to find substructures within the clusters. This kind of enhanced analysis can be performed with other methods [25]. Even when applying these other techniques, the ballistic search method is still necessary, since the cluster landscape has to be obtained in advance. There, the ballistic-search clustering is applied to guarantee that a ground-state landscape is sampled thermodynamically correct, see also [27].

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| L | $n_R$ | $r$ | $C$ | $n_{GS}$ | $s_0/k_B$ |
|---|---|---|---|---|---|
| 3 | 1000 | 1000 | 1.79(3) | $2.6(2)\times 10^1$ | 0.0842(16) |
| 4 | 1000 | $10^4$ | 2.58(6) | $2.1(1)\times 10^2$ | 0.0627(09) |
| 5 | 100 | $10^5$ | 4.3(3) | $1.3(4)\times 10^4$ | 0.0519(18) |
| 5 | 1000 | 3000 | 3.8(1) | $2.1(3)\times 10^4$ | 0.0560(08) |
| 6 | 1000 | 5000 | 6.6(3) | $1.3(3)\times 10^7$ | 0.0535(05) |
| 8 | 192 | $2\times 10^4$ | 24 (2) | $1.8(1.7)\times 10^{16}$ | 0.0520(07) |

Table 1: For each system size L ($d=3$): number $n_R$ of realizations, number $r$ of independent runs per realization, average number $C$ of clusters per realization, average ground state degeneracy $n_{GS}$ and the average entropy per spin $s_0$.

| L | $n_R$ | $r$ | $C$ | $n_{GS}$ | $s_0/k_B$ |
|---|---|---|---|---|---|
| 5 | 1000 | 1000 | 1.79(3) | $3.2(2)\times 10^1$ | 0.1041(17) |
| 7 | 1000 | $10^4$ | 2.58(6) | $5.4(5)\times 10^2$ | 0.0916(13) |
| 10 | 1000 | $10^4$ | 4.3(3) | $7.6(4)\times 10^5$ | 0.0868(10) |
| 14 | 1000 | 3000 | 3.8(1) | $1.5(9)\times 10^{12}$ | 0.0863(07) |
| 20 | 96 | 5000 | 6.6(3) | $5.1(4.9)\times 10^{25}$ | 0.0854(20) |

Table 2: For each system size L ($d=2$): number $n_R$ of realizations, number $r$ of independent runs per realization, average number $C$ of clusters per realization, average ground state degeneracy $n_{GS}$ and the average entropy per spin $s_0$.

| L | $n_R$ | $r$ | $C$ | $n_{GS}$ | $s_0/k_B$ |
|---|---|---|---|---|---|
| 3 | 1000 | 5000 | 2.99(9) | $2.7(2)\times 10^2$ | 0.0510(07) |
| 4 | 455 | 5000 | 5.2(3) | $9(1)\times 10^5$ | 0.0394(07) |
| 5 | 457 | 1000 | 9.9(5) | $7(7)\times 10^{14}$ | 0.0358(03) |
| 6 | 10 | 100 | 15(4) | $3(2)\times 10^{20}$ | 0.0319(16) |

Table 3: For each system size L ($d=4$): number $n_R$ of realizations, number $r$ of independent runs per realization, average number $C$ of clusters per realization, average ground state degeneracy $n_{GS}$ and the average entropy per spin $s_0$.

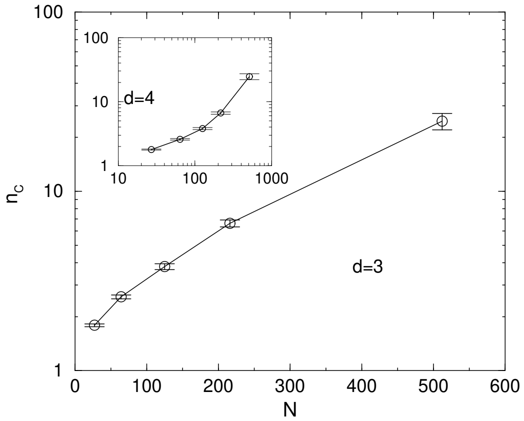


Figure 1: Number $n_C$ of ground-state clusters as a function of system size $N$ for $d = 3$. The inset shows the same data using a double-logarithmic scale. Lines are guide to the eyes only.

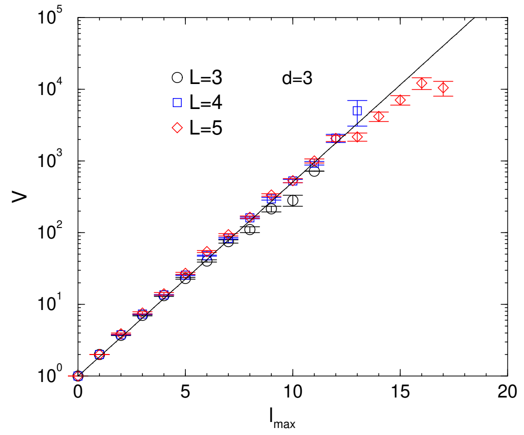


Figure 2: Average size $V$ of a cluster ($d = 3$) as a function of average dynamic number $l_{\mathrm{max}}$ of free spins (see text) for three-dimensional $\pm J$ spin glasses of system sizes $L = 3, 4, 5$, where all ground have been obtained. A $V = 2^{0.9 l_{\mathrm{max}}}$ dependence is found, indicated by a line.

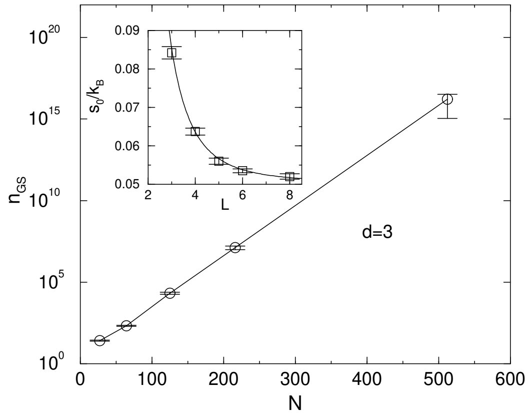


Figure 3: Number $n_{GS}$ of ground-states ($d = 3$) as a function of system size $N$. The number of states grows exponentially with the number of spins. Line is guide to the eyes only. The inset displays the ground-state entropy per spin as a function of $L$. The line shows a fit extrapolating $s_0$ to the infinite system which yields $s_0(\infty) = 0.0505(6)k_B$.

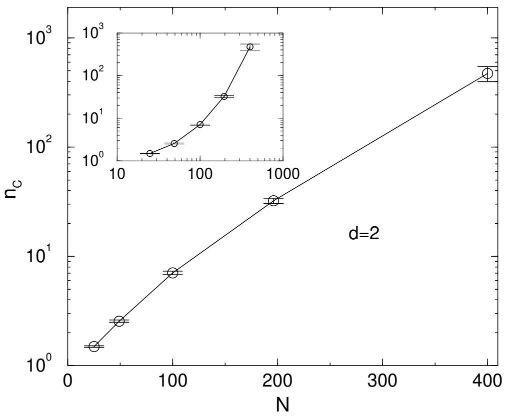


Figure 4: Number $n_C$ of ground-state clusters for two-dimensional $\pm J$ spin glasses as a function of system size $N$. The inset shows the same data using a double-logarithmic scale. Lines are guide to the eyes only.

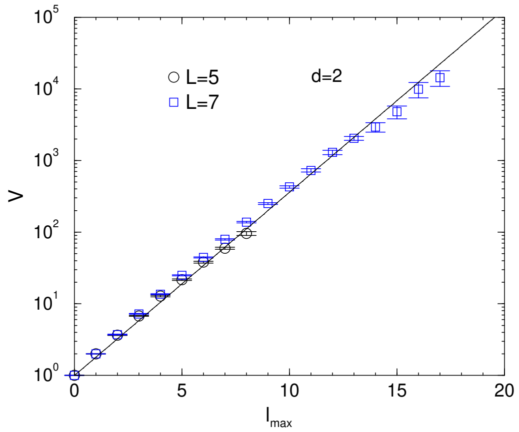


Figure 5: Average size $V$ of a cluster as a function of average dynamic number $l_{\max}$ of free spins (see text) for two-dimensional $\pm J$ spin glasses of system sizes $L = 5, 7$, where all ground have been obtained. A $V = 2^{0.85 l_{\max}}$ dependence is found, indicated by a line.

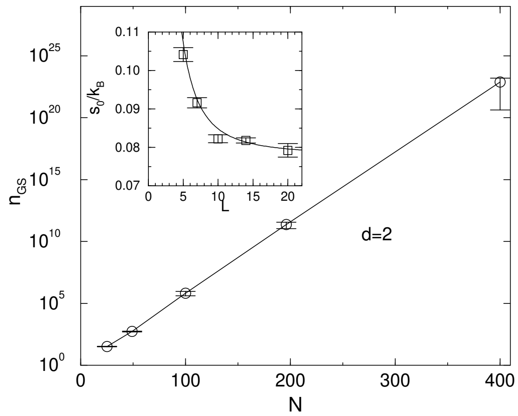


Figure 6: Number $n_{GS}$ of ground-states for two-dimensional $\pm J$ spin glasses as a function of system size $N$ (with $\alpha = 0.8$). The number of states grows exponentially with the number of spins. Lines are guide to the eyes only. The inset displays the ground-state entropy per spin as a function of $L$. The line shows a fit extrapolating $s_0$ to the infinite system which yields $s_0(\infty) = 0.078(5) k_B$.

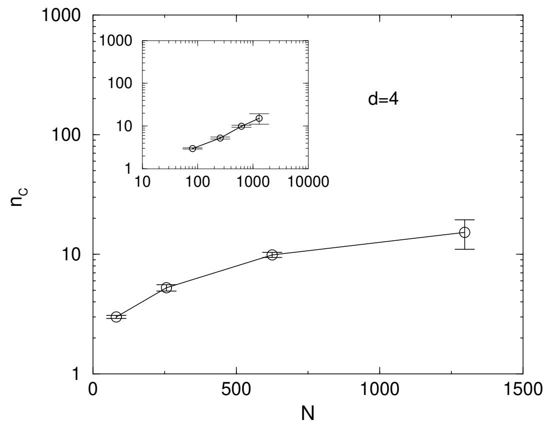


Figure 7: Number $n_C$ of ground-state clusters for four-dimensional $\pm J$ spin glasses as a function of system size $N$. The inset shows the same data using a double-logarithmic scale. Lines are guide to the eyes only.

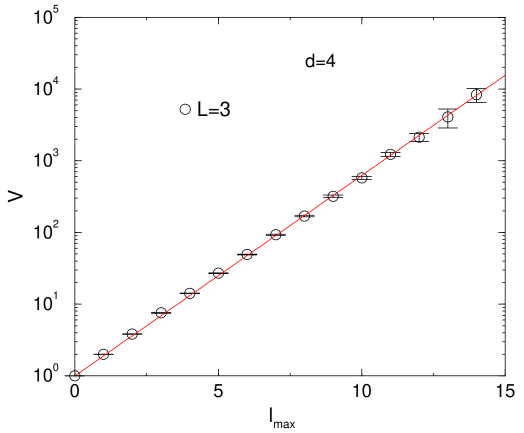


Figure 8: Average size $V$ of a cluster as a function of average dynamic number $l_{\max}$ of free spins (see text) for four-dimensional $\pm J$ spin glasses of system sizes $L = 3$, where all ground have been obtained. A $V = 2^{0.93 l_{\max}}$ dependence is found, indicated by a line.

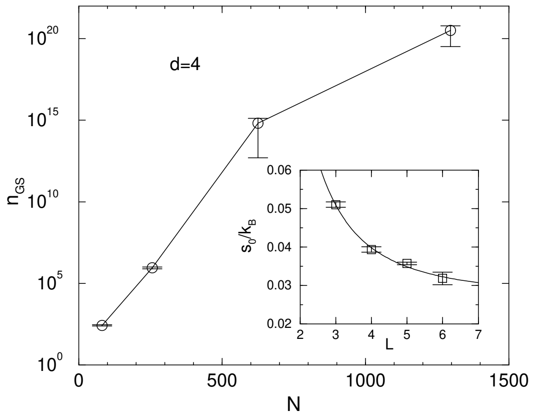


Figure 9: Number $n_{GS}$ of ground-states for four-dimensional $\pm J$ spin glasses as a function of system size $N$ (with $\alpha = 0.93$). The number of states grows exponentially with the number of spins. Lines are guide to the eyes only. The inset displays the ground-state entropy per spin as a function of $L$. The line shows a fit extrapolating $s_0$ to the infinite system which yields $s_0(\infty) = 0.027(5) k_B$.