Lower Critical Dimension of Ising Spin Glasses

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Exact ground states of two-dimensional Ising spin glasses with Gaussian and bimodal ($\pm J$) distributions of the disorder are calculated using a "matching" algorithm, which allows large system sizes of up to $N = 480^2$ spins to be investigated. We study domain walls induced by two rather different types of boundary-condition changes, and, in each case, analyze the system-size dependence of an appropriately defined "defect energy", which we denote by ΔE. For Gaussian disorder, we find a power-law behavior $\Delta E \sim L^\theta$, with $\theta = -0.266(2)$ and $\theta = -0.282(2)$ for the two types of boundary condition changes. These results are in reasonable agreement with each other, allowing for small systematic effects. They also agree well with earlier work on smaller sizes. The negative value indicates that two dimensions is below the lower critical dimension d_c. For the $\pm J$ model, we obtain a *different* result, namely the domain-wall energy saturates at a nonzero value for $L \to \infty$, so $\theta = 0$, indicating that the lower critical dimension for the $\pm J$ model *exactly* $d_c = 2$.

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Although most of the effort to understand spin glasses [1] has been concerned with the behavior in three dimensions, the situation in two dimensions is also not fully understood. For a Gaussian distribution of disorder, it is widely accepted that [2–4], $T_c = 0$ and two–dimensions is *below* the lower critical dimension d_c. However, for bimodal ($\pm J$) disorder, inconsistent results have been found. Using exact ground-state calculations of small systems ($N \le 30^2$) and introducing domain walls in the system by changing the boundary conditions, Kawashima and Rieger [5] also concluded that $T_c = 0$ and probably $d_c > 2$, though they could not rule out marginal behavior, i.e. $d_c = 2$. On the other hand, from Monte-Carlo (MC) simulations [6] on sizes $N \le 18^2$, it was claimed that $T_c > 0$. A calculation of domain-wall energies using a cluster MC method [7] for sizes $N \le 24^2$ indicated marginal behavior (i.e. $d_c = 2$), and this was interpreted as evidence for the finite-T transition claimed in Ref. [6]. Recently, Houdayer [8] used a more efficient cluster MC algorithm, which allows larger systems ($N \le 100^2$) to be studied down to low temperature $T = 0.1$. Houdayer finds that $T_c = 0$ but with an exponential divergence of the correlation length for $T \to 0$, which is consistent with $d_c = 2$, in agreement with earlier work by Saul and Kardar [9].

To try to clarify the situation, we calculate here ground-state domain-wall energies ΔE using an *exact* polynomial-time matching algorithm [12] for much larger system sizes ($N \le 480^2$) than in previous work. For the Gaussian model our results are consistent with earlier calculations and we find that $T_c = 0$ and $d_c > 2$. For the $\pm J$ model, we find marginal behavior expected at the lower critical dimension, and so conclude that $d_c = 2$.

The model consists of $N = L^2$ Ising spins $S_i = \pm 1$ on a square lattice with the Hamiltonian

$$\mathcal{H} = -\sum_{\langle i,j\rangle} J_{ij} S_i S_j \, , \tag{1}$$

where the sum runs over all pairs of nearest neighbors $\langle i,j\rangle$ and the J_{ij} are quenched random variables. Here, we consider two kinds of disorder distributions: (i) Gaussian with zero mean and variance one, and (ii) a bimodal distribution $J_{ij} = \pm 1$ with equal probability.

In greater than two dimensions, or in the presence of a magnetic field, the exact calculation of spin-glass ground states belongs to the class of NP-hard problems [10,11]. This means that only algorithms with exponentially increasing running time are known. However, for the special case of a planar system without magnetic field, e.g. a square lattice with periodic boundary conditions in at most one direction, there are efficient polynomial-time "matching" algorithms [12]. The basic idea is to represent each realization of the disorder by its frustrated plaquettes [13]. Pairs of frustrated plaquettes are connected by paths in the lattice and the weight of a path is defined by the sum of the absolute values of the coupling constants which are crossed by the path. A ground state corresponds the set of paths with minimum total weight, such that each frustrated plaquette is connected to exactly one other frustrated plaquette. This is called a minimum-weight perfect matching. The bonds which are crossed by paths connecting the frustrated plaquettes are unsatisfied in the ground state, and all other bonds are satisfied.

For the calculation of the minimum-weight perfect matching, efficient polynomial-time algorithms are available [14,15]. Recently, an implementation has been presented [16], where ground-state energies of large systems of size $N \le 1800$ were calculated. Here, an algorithm from the LEDA library [17] has been applied, which limits the system sizes to $N \le 480^2$, due to the restricted size of the main memory of the computers which we used.

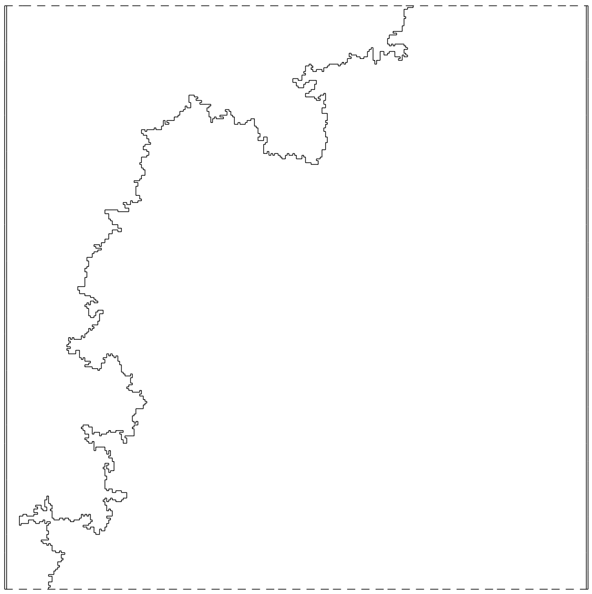

FIG. 1. A domain wall created in an $L = 320$ system with Gaussian interactions. Free boundary conditions are applied across the top and bottom edges (dashed lines) and periodic boundary conditions applied across the vertical edges (solid lines). The domain wall is induced by changing the periodic boundary conditions to antiperiodic, which is equivalent to changing the sign of the bonds which wrap round the system from left to right.

To study whether an ordered phase is stable at finite temperatures, the following procedure is usually applied [2–5,18–20]. First a ground state of the system is calculated, having energy E_0. Then the system is perturbed to introduce a domain wall and the new ground state energy, E_0^{pertb} is evaluated. Typically, the system initially has periodic boundary conditions in both directions, and the perturbation involves replacing periodic by anti-periodic boundary conditions in one direction. However we cannot apply the matching algorithm for boundary conditions which "wrap around" in both directions, so instead we have investigated two slightly different procedures:

(i) We choose boundary conditions which are free in one direction and periodic in the other (x-say). The periodic boundary conditions are then replaced by anti-periodic, which is equivalent to changing sign of the bonds which "wrap around" the system. This is very similar to the usual approach, except that the induced domain wall no longer has to have the same x coordinate at the top and bottom of the sample. An example of a domain wall formed in this way is shown in Fig. 1. We denote these as "P-AP" boundary condition changes. Note that the energy change can have either sign, and the distribution of values (obtained by repeating the calculation for many samples) is symmetric.

(ii) We apply free boundary conditions in both directions and compute the ground state. We then add extra bonds which wrap around the system in the x-direction say, and have a sign and strength such that they force the spins they connect to have the opposite relative orientation to that which they had in the original ground state. The new ground state is then computed and we remove from the ground state energy the contribution from the added strong bonds. The change in energy must then be positive. This procedure is similar to the "conjugate boundary conditions" chosen in Ref. [7]. Since free boundary conditions are still applied along the top and bottom edges, the domain walls look very similar to the one shown in Fig. 1. We denote this boundary condition change by "F-DW", since the initial boundary condition is **F**ree, and it is then changed to induce a **D**omain **W**all.

For a given sample, the domain-wall energy is given by

$$\delta E = E_0^{\mathrm{pertb}} - E_0 \,. \tag{2}$$

To study finite-size behavior we consider the "defect energy", ΔE, defined by

$$\Delta E = [\,|\delta E|\,]_{\mathrm{av}}, \tag{3}$$

where $[\cdots]_{\mathrm{av}}$ denotes an average over samples. (Of course the absolute value is not necessary for the F-DW boundary conditions since $\delta E \geq 0$.) It is expected that ΔE will vary as

$$\Delta E \sim L^{\theta}, \tag{4}$$

where θ is a stiffness exponent. For $\theta < 0$ an ordered phase is only stable at $T = 0$ and the correlation length ξ diverges [3] for $T \to 0$ as $\xi \sim T^{1/\theta}$. For $\theta > 0$ an ordered phase also occurs for $T > 0$. At the lower critical dimension, one has $\theta = 0$ and expects an exponential divergence of the correlation length.

Domain-wall energies obtained in the way described above were obtained for systems with Gaussian and $\pm J$ disorder, for both F-DW and P-AP boundary condition changes. Sizes up to $N = 480^2$ were considered, and, for each size, the result was average over up to 40000 samples. Table I shows the number of samples for each system size and type of system.

type	$L \leq 160$	$L = 240$	$L = 320$	$L = 480$
$\pm J$ P-AP	40000	45000	24000	5000
$\pm J$ F-DW	10000	5500	4500	-
Gaussian P-AP	10000	2800	3700	-
Gaussian F-DW	10000	700	-	-

TABLE I. The number of samples for each type of disorder and system size.

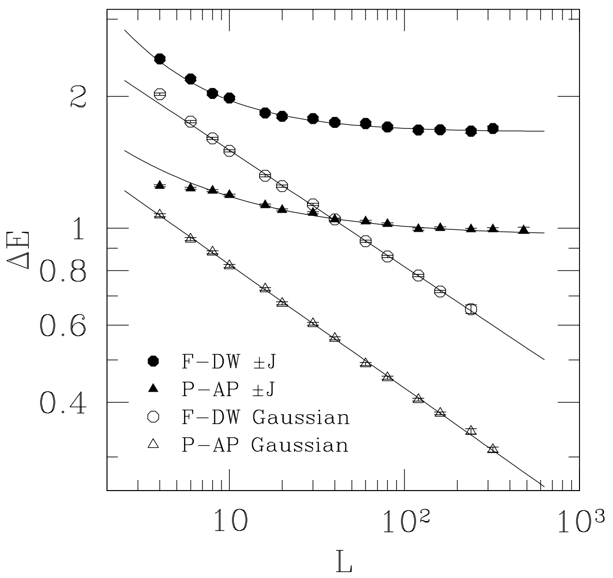

FIG. 2. The defect energy ΔE as a function of system size for the different types of disorder and boundary condition changes. The lines are fits (for $L \geq 8$) to the form $\Delta E = \Delta E_\infty + bL^\theta$. The following fit parameters were obtained: $\Delta E = 0.96$, $b = 0.99$, $\theta = -0.65$ ($\pm J$ P-AP); $\Delta E = 1.66$, $b = 2.93$, $\theta = -0.99$ ($\pm J$ F-DW); $b = 1.58$, $\theta = -0.282$ ($\Delta E \equiv 0$, Gaussian P-AP); $b = 2.78$, $\theta = -0.266$ ($\Delta E \equiv 0$, Gaussian F-DW).

In Fig. 2 the defect energies are shown as a function of system size. We first discuss the results for the Gaussian distribution. The domain-wall energy clearly decreases with a power law $\Delta E \sim L^\theta$ with

$$\begin{aligned} \theta &= -0.282 \pm 0.002 \qquad (\mathrm{P-AP}) \\ \theta &= -0.266 \pm 0.002 \qquad (\mathrm{F-DW}). \end{aligned} \tag{5}$$

For the goodness of the fits [21] we obtain $Q = 0.48$ and $Q = 0.62$ for the P-AP and F-DW boundary conditions respectively. In addition, we tried fits of the form $\Delta E = \Delta E_\infty + bL^\theta$, obtaining $\Delta E_\infty = 0.04 \pm 0.02$ ($Q = 0.48$) for P-AP boundary conditions and $\Delta E_\infty = -0.03 \pm 0.07$ ($Q = 0.54$) for F-DW boundary conditions. Hence, we cannot rule out that the domain-wall energy converges to a small nonzero value for $L \to \infty$ but this seems unlikely.

The result of the exponent for the P-AP boundary conditions is compatible with the value of $\theta = -0.281(2)$ found [4] for full periodic boundary conditions and system sizes $L \leq 30$. The value of the F-DW case disagrees with the value for the P-AP case by more than the error bars, but the difference is quite small. We expect that this is due to systematic corrections to scaling and that asymptotically the same result would be obtained, i.e. the same exponent would be obtained for any reasonable definition of the defect energy. Our result for the F-DW boundary condition differs significantly from the value $\theta = -0.20$ obtained [7]. The reason for the discrepancy may be that Ref. [7] studied smaller sizes, $L \leq 24$, and used a Monte-Carlo method to determine the ground states instead of an exact algorithm. Nevertheless, to our knowledge, all results obtained for the Gaussian model are compatible with $T_c = 0$.

Next we turn to the results for the $\pm J$ model. The difference between these results and those for the Gaussian model, both of which are shown in Fig. 2, is quite striking. Whereas the data for the Gaussian model clearly tends to zero with a power law for $L \to \infty$, the data for the $\pm J$ distribution appears to tend to a non-zero value. Fits of the same form as for the Gaussian model give $\Delta E_\infty = 0.96 \pm 0.01$ ($Q = 0.23$) for the P-AP boundary conditions and $\Delta E_\infty = 1.66 \pm 0.01$ ($Q = 0.50$) for the F-DW boundary conditions. Fits in which ΔE_∞ was fixed to zero were also tried but gave ridiculously low probabilities: $Q = 2 \times 10^{-27}$ and $Q = 2 \times 10^{-17}$ respectively. For the case with free boundary conditions in the x-direction, a saturation has been observed before [7] for small system sizes $L \leq 24$, using a Monte-Carlo method to find ground states, rather than an exact algorithm.

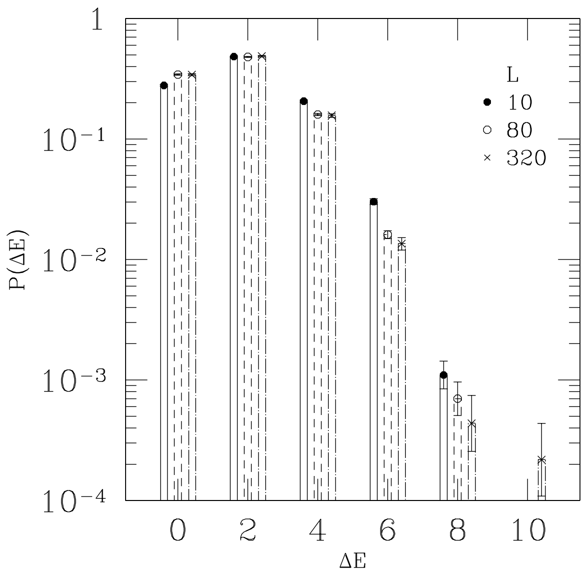

FIG. 3. Distribution $P(\Delta E)$ of domain-wall energies for the $\pm J$ distribution with F-DW boundary conditions. The points for $L = 320$ and $L = 10$ are slightly displaced on the x-axis for visibility.

To show the effect of saturation into more detail, Fig. 3 displays the distribution of the domain-wall energies, with a logarithmic vertical scale, for the $\pm J$ distribution with F-DW boundary conditions. For this distribution, only even integer values of ΔE are found. Except for the largest values of ΔE, where the weights are very small, the weights of the discrete bins are almost the same for $L = 80$ and 320 indicating that the whole distribution converges for large L. The probability $P(0)$ that the domain-wall energy is exactly zero is displayed in Fig. 4. $P(0)$ seems to converge towards a finite value less than unity for both boundary conditions.

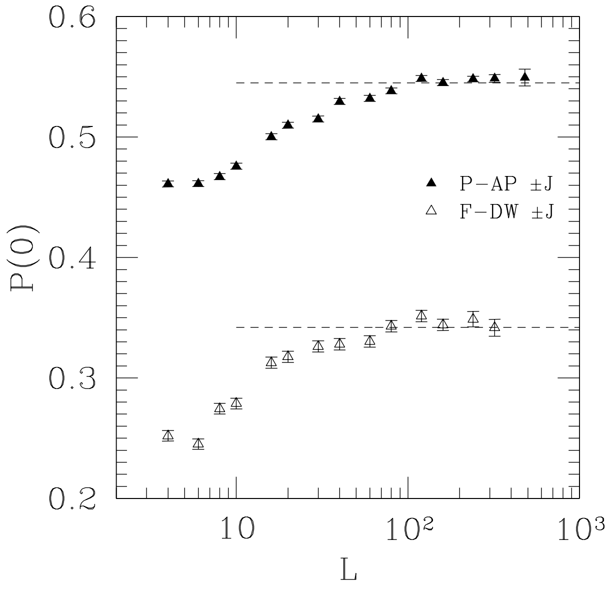

FIG. 4. The probability $P(0)$ that the domain-wall energy is exactly zero for the $\pm J$ model as a function of system size. The lines are guides to the eye.

From this data we deduce that $\theta = 0$ for the $\pm J$ distribution, so the lower critical dimension is $d_c = 2$ exactly. The most likely scenario at finite temperature is that the correlation length diverges exponentially as $T \to 0$, perhaps like $\exp(C/T)$ or $\exp(C/T^2)$. Detailed Monte Carlo simulations [8] and the work of Saul and Kardar [9] indicate that the former is the case. The other possibility is a finite-T Kosterlitz-Thouless type of transition without true long range order below T_c, as claimed by Ref. [6]. However, the finite value for ν obtained in Ref. [6] seems incompatible with this picture. The reason for this discrepancy is probably that the system sizes were quite small, $N \leq 18^2$.

To conclude, we have studied the defect energy of the two-dimensional Ising spin glass using much larger system sizes than before, up to $N = 480^2$. For Gaussian disorder we find that a spin-glass phase is not stable at $T > 0$, and that $d_c > 2$, in agreement with earlier findings for much smaller systems. For the $\pm J$ model, the absolute value of the domain-wall energy saturates at a nonzero value for $L \to \infty$, indicating that the lower critical dimension is exactly $d_c = 2$, in agreement with Refs. [8] and [9]. It is quite striking and perhaps surprising that the $\pm J$ model, which has a large ground state degeneracy, actually has *more* order at very low-T than the Gaussian distribution, which has a unique ground state.

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