Ferromagnetic-spin glass transition in four-dimensional random-bond Ising model

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The four-dimensional $\pm J$ random-bond Ising model is studied using ground-state calculations. System sizes up to $N = 6^4$ spins are considered. Here it is found that the ferromagnetic-spin glass transition occurs at a critical concentration $p_c = 0.28(1)$ of the antiferromagnetic bonds, which is comparable to values found previously by high-temperature series expansions. The transition is characterized by a correlation-length exponent $\nu = 1.0(1)$ and an order-parameter exponent $\beta = 0.4(1)$. Thus, this transition is in a different universality class from four-dimensional bond percolation, where $\nu = 0.678(50)$ and $\beta = 0.639(20)$.
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INTRODUCTION

The study of how order arises in nature is in the center of interest in several areas of physics, especially in thermodynamics and statistical physics. A special focus is on the question, how order can emerge, even in or due to the presence of quenched disorder. One type of system, which has attracted a lot of attention during the last decades, are spin glasses[1].

In this work, the transition from one ordered phase, the ferromagnetic phase, to another ordered phase, the spin-glass phase, is studied for a four-dimensional random-bond Ising model. The model is studied by means of ground-state calculations, using the *genetic cluster-exact approximation (CEA)* method. This approach has the advantage that one does not encounter ergodicity problems or critical slowing down like when using algorithms which are based on Monte-Carlo methods. To the author's knowledge, there are no numerical studies of the four-dimensional random-bond Ising model (except at $p = 0.5$). However, for two[2–4] and three[5] dimensions, the random-bond Ising model has been studied numerically at $T = 0$ already, resulting in critical values $p_c^{\rm sq} = 0.10$ for square lattices respectively $p_c^{cub} = 0.22$ for cubic lattices.

The model treated here consists of $N = L^4$ Ising spins $\sigma_i = \pm 1$ on a hypercubic lattice. The Hamiltonian of this system is given by

$$H \equiv -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j \qquad (1)$$

where $\langle \ldots \rangle$ denotes a sum over pair of nearest neighbors. Systems with quenched disorder of the bonds $J_{ij} = \pm 1$ are studied. The ferromagnetic-spin glass transition is driven by increasing the fraction p of antiferromagnetic ($J_{ij} = -1$) (AF) bonds. For low concentration p, the system is ferromagnetically ordered, while for intermediate values of p, spin-glass order arises. For very high concentration, the system shows AF ordering.

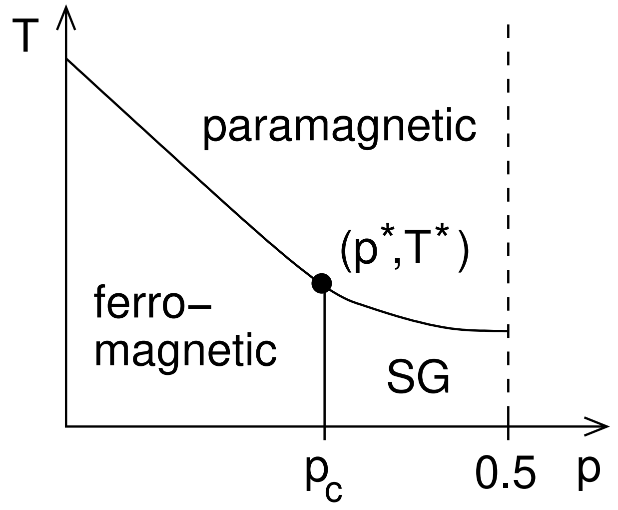

FIG. 1. Schematic phase diagram of the four-dimensional random-bond model. The diagram is symmetric with respect to $p = 0.5$, with an antiferromagnetically ordered phase for large p. SG denotes the spin glass phase. The multicritical point is denoted by (p^*, T^*).

In previous work, the model has been studied using high-temperature series expansions[6–8]. In Fig. 1 the schematic phase diagram is presented. The values previously obtained for the concentration at the multicritical point are $p^* \approx 0.3$ (Ref. 6), $p^* = 0.28(1)$ (Ref. 7) and $p^* = 0.290(1)$ (Ref. 8). The value of p_c has not been determined so far. It is only known that the line connecting the multicritical point (p^*, T^*) with $(p = p_c, T = 0)$ starts vertically[9,10] at the multicritical point. Therefore, $p_c \approx p^*$ can be expected. This is consistent with a recent conjecture[11] for the critical concentration of a d-dimensional ($d > 2$) hypercubic random-bond system: $p_c(d) = 0.5(1 - 1/d)^2$, which gives $p_c \approx 0.281$ in four dimensions. The value found in the present work is $p_c = 0.28(1)$.

Beyond the determination of p_c, it is interesting to determine the critical exponents describing the behavior around the critical point. Very often, phase transitions can be explained by means of percolation transitions[12]. Thus, always the question arises whether the values of the exponents are equal to the exponents of the corresponding percolation transitions. Phase transitions induced by percolation can be found even in very unphysical models. For example, recently the onset of replica-symmetry breaking[13] in the solution of a combinatorial optimization problem, the vertex-cover problem[14], has been related[15] to a new type of percolation transition for random graphs. In the case of the random-bond model, for two[8] and three[16] dimensions the results obtained at the multicritical point are very close to the results for bond percolation[17], while in four dimensions the results[8,17] disagree. For the ground-state calculations, the exponents obtained in two[4] dimensions are also compatible with percolation, while for three[5] dimensions some deviations occur. In this paper it is shown, that also at $T = 0$ the four-dimensional result is clearly not compatible with a percolation-like transition.

The rest of the paper is organized as follows: Next, the algorithm to generate spin-glass ground states in four dimensions is briefly explained. Then, the results describing the ferromagnetic-spin glass transition are presented and finally a summary is given.

ALGORITHM

From the computational point of view, the calculation of spin-glass ground states is very demanding, because it belongs to the class of the NP-hard problems[18,19]. This means that only algorithms are available, for which the running time on a computer increases exponentially with the system size. Only for the special case of a planar system without magnetic field, e.g. a square lattice with periodic boundary conditions in at most one direction, exist efficient polynomial-time "matching" algorithms[20].

For higher dimensions, one must rely on exact methods like branch-and-bound[21,22] or branch-and-cut[23,24], which are able to treat only small systems. For that reason, recently some heuristics have been introduced. By using a hierarchical approach[25] one can calculate true ground states in four dimensions up to size $L = 5$, while for $L = 6$ it is not clear whether true ground states were found[26].

The method applied here[27], is able to calculate ground-states in four dimensions up to size $L = 7$. The technique is based on a special genetic algorithm[28,29] and on cluster-exact approximation[30] which is an optimization method designed especially for spin glasses. Now a brief description of the techniques are given.

Genetic algorithms are biologically motivated. An optimal solution is found by treating many instances of the problem in parallel, keeping only better instances and replacing bad ones by new ones (survival of the fittest). The genetic algorithm starts with an initial population of M_i randomly initialized spin configurations (= *individuals*), which are linearly arranged in a ring. Then $\nu \times M_i$ times two neighbors from the population are taken (called *parents*) and two *offspring* are created using the so called triadic crossover[31]. Then a mutation with a rate of p_m is applied to each offspring, i.e. a fraction p_m of the spins is reversed. Next, for both offspring the energy is reduced by applying the CEA. This algorithm constructs iteratively and randomly a non-frustrated cluster of spins, whereas spins with many unsatisfied bonds are more likely to be added to the cluster. The non-cluster spins act like local magnetic fields on the cluster spins. For the spins of the cluster, an energetic minimum state can be calculated in polynomial time by using graph-theoretical methods[32–34]: an equivalent network is constructed[35], the maximum flow is calculated[36,37] and the spins of the cluster are set to the orientations leading to a minimum in energy. This minimization step is performed $n_{\min}$ times for each offspring.

Afterwards each offspring is compared with one of its parents. The pairs are chosen in the way that the sum of the phenotypic differences between them is minimal. The phenotypic difference is defined here as the number of spins where the two configurations differ. Each parent is replaced if its energy is not lower (i.e. better) than the corresponding offspring.

After this creation of offspring has been performed $\nu \times M_i$ times, the population is halved: From each pair of neighbors the configuration which has the higher energy is eliminated. If not more than 4 individuals remain, the process is stopped and the best individual is taken as result of the calculation.

The whole algorithm is performed n_R times and all final configurations which exhibit the lowest energy are stored, resulting in n_g statistical independent ground-state configurations. By comparison with an exact method, it was shown[38] that genetic CEA, with an appropriate choice of the parameters (M_i, ν, $n_{\min}$, p_m), indeed calculates true ground states.

The probability that a certain ground-state configuration is found by this method is not equal for all ground states[39], i.e. the algorithm imposes a bias. In this work, the magnetization is the main quantity of interest. To test, how large the influence of this bias is, for $L = 6$, $p = 0.27$, where fluctuations of the magnetization are the largest, 64 realizations were considered. For each realization, (biased) ground states were generated using 200 independent runs. Next, for each realization a (thermodynamically correct) set of ground states was generated in such a way that each configuration contributes with the same probability[40]. For both sets of ground states the magnetization was evaluated. For the biased set an average value of $m = 0.501(11)$ was obtained, while the correct thermodynamic result is $m = 0.504(12)$. This shows that the influence of the bias on the result of the magnetization is small. Please note that that is in contrast to other quantities like the distribution $P(q)$ of overlaps: The overlap $q^{\alpha\beta}$ between two independent ground states $\{\sigma_i^\alpha\}$, $\{\sigma_i^\beta\}$ is given by $q = \frac{1}{N}\sum_i \sigma_i^\alpha \sigma_i^\beta$. When measuring the fraction $x(0.5) = \int_{-0.5}^{0.5} P(q)dq$ of small overlaps, one obtains an average value $x(0.5) = 0.051(7)$ for the biased set of states, while the from the correct thermodynamic treatment one obtains quite a different result 0.014(8).

Hence, when restricting the measurement to quantities depending solely on the magnetization, it is sufficient to use the biased data. This allows much shorter running times, because only few ground states per realization are needed.

RESULTS

With the genetic cluster-exact approximation, it it possible[27] to obtain ground-states for four-dimensional systems up to $L = 7$. Since the study of the transition involves the calculation of a Binder cumulant, it is necessary to average over many samples of the disorder. For that reason, only sizes $L \leq 6$ were considered in this work. A mutation rate of $p_m = 0.05$ was used. The other simulation parameters are shown in Tab. I.

L	M_i	ν	$n_{\min}$	τ (sec)	N_L
3	16	4	4	3	10^5
4	16	4	4	14	15000
5	256	6	10	4800	13000
6	256	6	10	7300	20500
7	512	12	20	14000	-

TABLE I. Simulation parameters: L = system size, M_i = initial size of population, ν = average number of offspring per configuration, $n_{\min}$ = number of CEA minimization steps per offspring, τ = typical computer time per ground state on a 80MHz PPC601, N_L = total number of realizations of the random variables.

Systems with concentrations of the antiferromagnetic bonds in the range $p \in [0, 0.32]$ were treated. For each realization $n_R = 5$ independent runs were performed. The resulting ground-state energy as a function of the parameter p is shown in Fig. 2 for different system sizes. For small concentrations p, the ground state is mainly ferromagnetic. It follows that all the AF bonds are not satisfied, so the ground state energy increases linearly with p like $e(p) \approx -4 + 8p$. For larger concentrations, the ground-state energy approaches the $p = 0.5$ limit. With increasing L the ground state energy decreases, because the periodic boundary conditions impose less constraints on the system. For $L \to \infty$ and $p = 0.5$ a ground state energy of $e_\infty(0.5) = -2.095(1)$ was found[27].

From Fig. 2 is clear that the energy as function of concentration is not a suitable quantity for determining the critical concentration p_c, where the ferromagnetic behavior disappears. For this purpose the Binder cumulant[41,42]

$$g(p, L) \equiv \frac{1}{2}\left(3 - \frac{[\langle M^4\rangle]_J}{[\langle M^2\rangle]_J^2}\right) \tag{2}$$

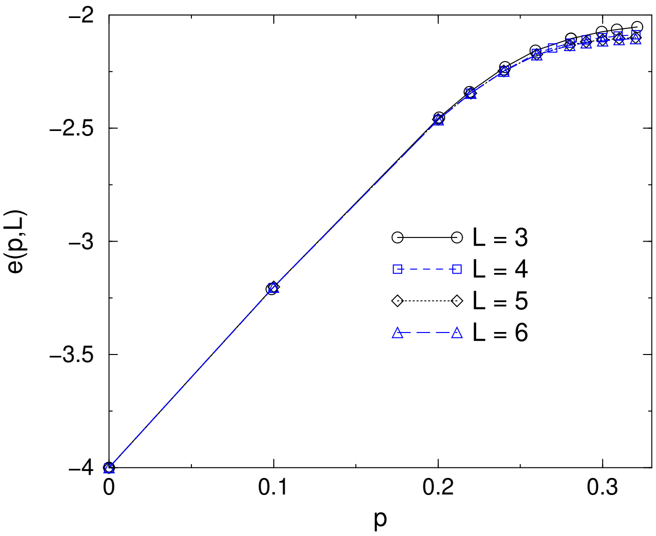

FIG. 2. Ground-state energy $e(p, L)$ as a function of the AF-bond concentration p for system sizes $L = 3, 4, 5, 6$. Error bars are much smaller than symbol sizes. Lines are guides for the eyes only.

for the magnetization $M \equiv \frac{1}{N}\sum_i \sigma_i$ is evaluated. The average $\langle \ldots \rangle$ denotes the average over different ground states of a realization, while $[\ldots]_J$ is the average over the disorder. In Fig. 3 the Binder cumulant is shown for $L = 3, 4, 5, 6$. To keep the figure clear, only the largest error bar is shown. All curves intersect near $p_c = 0.28 \pm 0.01$. Because of the huge numerical effort, the simulations are restricted to small system sizes, which prevents a more accurate result.

Only few studies concerning the four-dimensional random-bond model have been performed before. Using high-temperature series expansions for the location of the multicritical point (p^*, T^*), values of $p^* \approx 0.3$ (Ref. 6), $p^* = 0.28(1)$ (Ref. 7) and $p^* = 0.290(1)$ (Ref. 8) were determined. These results compare well with the value of p_c obtained here, indicating that the the line connecting $(p_c, 0)$ with (p^*, T^*) is vertical, or nearly so, as expected[9,10].

For the Binder cumulant the following finite-size scaling relation is assumed[42]

$$g(p, L) = \tilde{g}(L^{1/\nu}(p - p_c)) \,. \tag{3}$$

By plotting $g(p, L)$ against $L^{1/\nu}(p - p_c)$ with correct parameters p_c, ν the datapoints for different system sizes should collapse onto a single curve near $(p - p_c) = 0$. The best results were obtained for $p_c = 0.28$ and $1/\nu = 1.0$. In Fig. 4 the resulting scaling plot is shown. It is possible to change the value of ν in a wide range without large effects on the scaling plot. Thus, here a value of $\nu = 1.0(2)$ is estimated. This is compatible with the value $\nu = 0.8(1)$ found by a high-temperature series expansion[8] at the multicritical point. Although

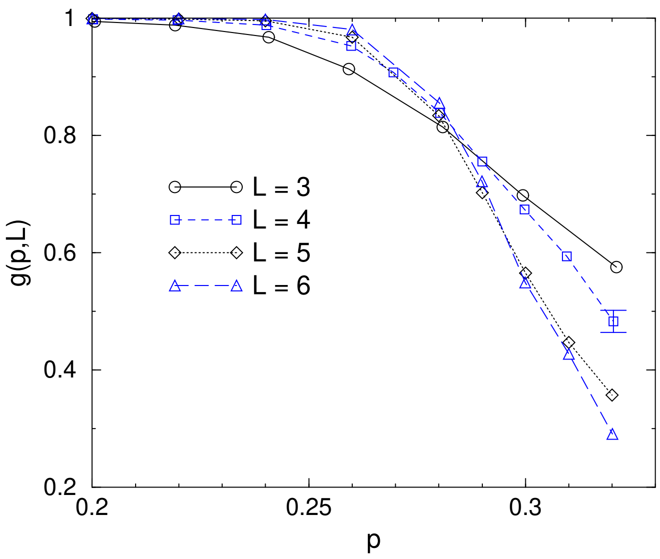

FIG. 3. Binder cumulant $g(p, L)$ of the ground-state magnetization as a function of the AF-bond concentration p for system sizes $L = 3, 4, 5, 6$. For clarity, only the largest error bar occurring in the data is shown. Lines are guides for the eyes only.

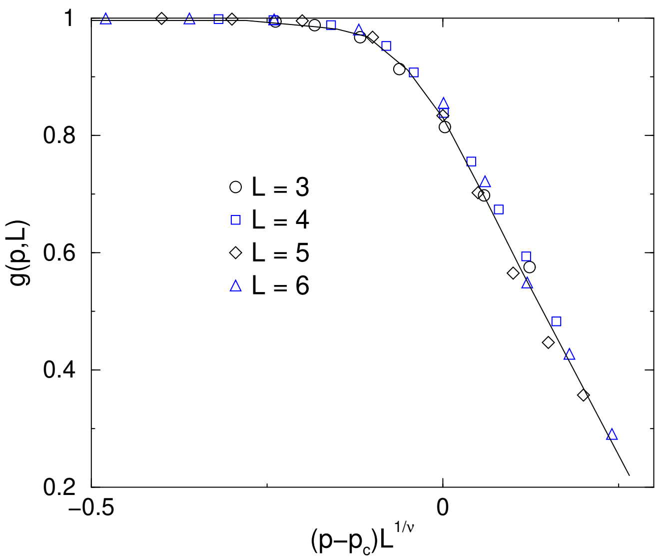

FIG. 4. Scaling plot of the Binder cumulant as a function of $(p - p_c)L^{1/\nu}$ with $p_c = 0.28$ and $\nu = 1.0$. The series of segments is a guide for the eyes only.

there is no direct correspondence between the $T = 0$ ferromagnetic-spin glass transition and the finite temperature ($p = 0.5$) paramagnetic-spin glass transition at $T_c = 2.03(3)$, it is remarkable that by Monte-Carlo simulations[43] of four-dimensional spin glasses near T_c a similar value of $\nu = 1.0(1)$ was found. Finally, it should be pointed out, that the result given above is not compatible with the corresponding exponent for four-dimensional bond percolation[17] of $\nu = 0.678(50)$.

In Fig. 5 the behavior of the average absolute value of the magnetization $m \equiv [\langle|M|\rangle]_J$ is shown as a function of p for different system sizes. This quantity has the standard finite-size scaling form[44]

$$m(p, L) = L^{-\beta/\nu}\tilde{m}(L^{1/\nu}(p - p_c)) \,. \qquad (4)$$

By plotting $L^{\beta/\nu}m(p, L)$ against $L^{1/\nu}(p-p_c)$ with correct parameters p_c, β, ν the datapoints for different system sizes should collapse onto a single curve near $(p-p_c) = 0$. The best result was obtained using $p_c = 0.28$, $1/\nu = 1.0$ and $\beta/\nu = 0.4$. It is shown in Fig. 6 for $L = 3, 4, 5, 6$. From variations of the value β/ν, the value of the exponent $\beta = 0.4(1)$ is estimated. Again, this values significantly differs from the exponent $\beta = 0.693(20)$ found[17] for the order parameter of the four-dimensional bond-percolation.

SUMMARY

The ferromagnetic-spin glass transition of the four-dimensional random-bond Ising model was studied at $T = 0$ using ground-state calculations. The genetic cluster-exact approximation method was applied. Because of the high computational effort, only small systems of size $N \leq 6^4$ could be studied. Since the ground-state problem is NP-hard, it is very unlikely that significantly larger sizes can be studied in the near future.

By evaluating the Binder cumulant, a critical concentration of $p_c = 0.28(1)$ was found, which is comparable to values which were obtained by high-temperature series expansions.

Using a finite-size scaling analysis, the critical exponents for the divergence of the correlation length $\nu = 1.0(1)$ and for the magnetization $\beta = 0.4(1)$ were determined. Hence, the transition is clearly in a different universality class than four-dimensional bond percolation.

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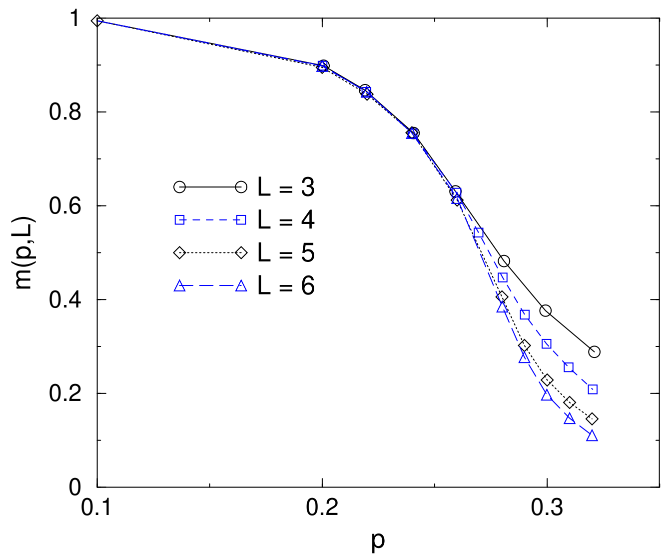

FIG. 5. Average absolute value $m(p, L)$ of ground-state magnetization as a function of the AF-bond concentration p for system sizes $L = 3, 4, 5, 6$. All error bars are much smaller than symbol size. Lines are guides for the eyes only.

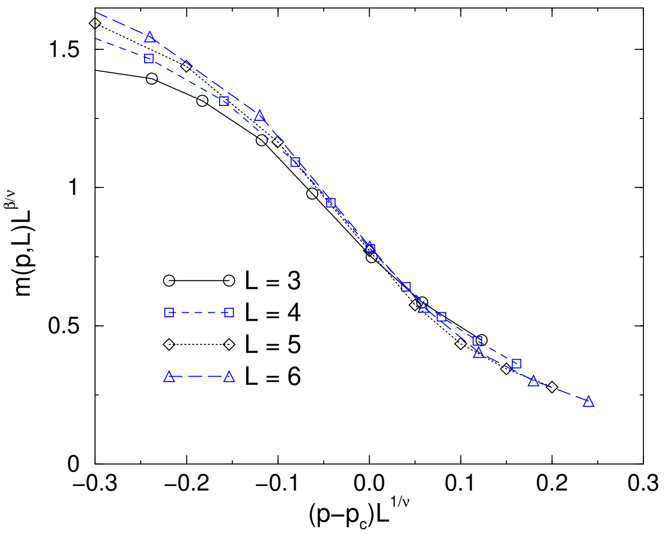

FIG. 6. Scaling plot of the rescaled absolute value $m(p, L)L^{\beta/\nu}$ of the magnetization as a function of $(p-p_c)L^{1/\nu}$ with $p_c = 0.28$, $\nu = 1.0$ and $\beta = 0.4$. Lines are guides for the eyes only.

[1] K. Binder and A.P. Young, Rev. Mod. Phys. **58**, 801 (1986); M. Mezard, G. Parisi, M.A. Virasoro, *Spin glass theory and beyond*, (World Scientific, Singapore 1987); K.H. Fisher and J.A. Hertz, *Spin Glasses*, (Cambridge University Press, Cambridge 1991); A.P. Young (ed.), *Spin glasses and random fields*, (World Scientific, Singapore 1998)

[2] J. Bendisch, J. Stat. Phys. **67**, 1209 (1992)

[3] J. Bendisch, Physica A **202**, 48 (1994)

[4] N. Kawashima and H. Rieger, Europhys. Lett. **39**, 85 (1997)

[5] A.K. Hartmann, Phys. Rev. B **59** , 3617 (1999)

[6] V.T. Rajan and P.S. Riseborough, Phys. Rev. B **27**, 532 (1983)

[7] R. Fisch, Phys. Rev. B **44**, 652 (1991)

[8] R.R.P. Singh and J. Adler, Phys. Rev. B **54**, 364 (1996)

[9] H. Nishimori, Prog. Theor. Phys. **66**, 1169 (1981); **76**, 305 (1986)

[10] P. Le Doussal and A.B. Harris, Phys. Rev. Lett. **61**, 625 (1988)

[11] J. Bendisch, private communication

[12] D. Stauffer and A. Aharony, Introduction to Percolation Theory, (Taylor and Francis, London) (1994)

[13] M. Mezard, G. Parisi, M.A. Virasoro, *Spin glass theory and beyond*, (World Scientific, Singapur 1987)

[14] M. R. Garey and D. S. Johnson, *Computers and intractability* (Freeman, San Francisco, 1979)

[15] M. Bauer, O. Golinelli, preprint cond-mat/0102011 (2000)

[16] R.R.P. Singh, Phys. Rev. Lett. **67**, 899 (1991)

[17] J. Adler, Y. Meir, A. Aharony, and A.B. Harris, Phys. Rev. B **41**, 9183 (1990)

[18] F. Barahona, J. Phys. A **15**, 3241 (1982).

[19] A.K. Hartmann and H. Rieger, *Optimization Algorithms in Physics*, Wiley-VCH, Berlin, to be published 2001.

[20] I. Bieche, R. Maynard, R. Rammal, and J.P. Uhry, J. Phys. A **13**, 2553 (1980).

[21] A. Hartwig, F. Daske and S. Kobe, Comp. Phys. Commun. **32** 133 (1984)

[22] T. Klotz and S. Kobe, J. Phys. A: Math. Gen. **27**, L95 (1994)

[23] C. De Simone, M. Diehl, M. Jünger, P. Mutzel, G. Reinelt and G. Rinaldi, J. Stat. Phys. **80**, 487 (1995)

[24] C. De Simone, M. Diehl, M. Jünger, P. Mutzel, G. Reinelt and G. Rinaldi, J. Stat. Phys. **84**, 1363 (1996)

[25] J. Houdayer and O. C. Martin, Phys. Rev. Lett. **83**, 1030 (1999)

[26] J. Houdayer and O. C. Martin, preprint cond-mat/0105617 (2001)

[27] A.K. Hartmann, Phys. Rev. E **60**, 5135 (1999)

[28] K.F. Pál, Physica A **223**, 283 (1996)

[29] Z. Michalewicz, *Genetic Algorithms + Data Structures = Evolution Programs*, (Springer, Berlin 1992)
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[30] A.K. Hartmann, Physica A **224**, 480 (1996)

[31] K.F. Pál, Biol. Cybern. **73**, 335 (1995)

[32] J.D. Claiborne, *Mathematical Preliminaries for Computer Networking*, (Wiley, New York 1990)

[33] W. Knödel, *Graphentheoretische Methoden und ihre Anwendung*, (Springer, Berlin 1969)

[34] M.N.S. Swamy and K. Thulasiraman, *Graphs, Networks and Algorithms*, (Wiley, New York 1991)

[35] J.-C. Picard and H.D. Ratliff, Networks **5**, 357 (1975)

[36] J.L. Träff, Eur. J. Oper. Res. **89**, 564 (1996)

[37] R.E. Tarjan, *Data Structures and Network Algorithms, (Society for industrial and applied mathematics*, Philadelphia 1983)

[38] A.K. Hartmann, Phys. Rev. E **59** , 84 (1999)

[39] A. Sandvik, Europhys. Lett. **45**, 745 (1999); A.K. Hartmann, Europhys. Lett. **45**, 747 (1999)
[40] A.K. Hartmann, Eur. Phys. J. B **13**, 539 (2000)
[41] K. Binder, Z. Phys. B **43**, 119 (1981)
[42] R.N. Bhatt and A.P. Young, Phys. Rev. Lett. **54**, 924 (1985); Phys. Rev. B **37**, 5606 (1988)
[43] E. Marinari and F. Zuliani, J. Phys. A **32**, 7447 (1999)
[44] K. Binder and D.W. Heermann, *Monte Carlo Simulations in Statistical Physics*, (Springer, Berlin 1988)