Intuitive understanding of $T \to 0$ behavior of 2d spin glasses via renormalization-group analysis

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In a recent paper [1], Thomas Joerg and Florent Krzakala study two-dimensional Ising spin glasses via a Migdal-Kadanoff Renormalization Group (MKRG) study.

Let us start this *News and Perspective* contribution by briefly discussing the main terms of the first sentence: Spin glasses are models for disordered magnets (like an iron-gold alloy), frequently studied in statistical physics [2, 3, 4, 5]. Its Ising version, where each spin may only take one of two possible orientations, is conceptually a very simple version. However, due to the appearance of disorder and frustration, hence competing interactions which cannot be satisfied all together, a complex behavior emerges at low temperatures. Despite more than three decades of intensive research, many properties of spin glasses, especially in finite dimensions, are still not well understood.

The basic idea of the (real-space) renormalization-group (RG) approach [6, 7, 8] is very simple: a physical system, when studied at different (length) scales, should look the same — if the scales are sufficiently large enough. The widespread success of many methods over many fields based on this simple observation was highlighted by the Nobel Price awarded in 1982 to Kenneth G. Wilson for, among other achievements, his contributions to the understanding of phase transitions using RG theory. RG is most simple explained by a block spin transformation of Leo Kadanoff [9], where in a magnetic spin system one joins groups of spins to form block spins. Correspondingly, the effective interactions between the block spins are calculated from the interactions of the microscopical spins. Hence, the interactions of a model are changed under the block transformation. Usually one ignores higher order interactions, which are usually created during a block transformation. Hence, in this case, the approach is an approximation.

For a system with quenched disorder, the interactions are characterized by distributions. Hence, when applying RG transformations, one transforms distributions making RG a *functional* transformation in this case, usually also involving further approximations. Joerg and Krzakala use a specific transformation, the MKRG [10, 11]. Here, for pedagogical reasons, a short explanation is given: Within the approach, instead of merging groups of spins one merges groups of bonds. The approach can be best understand by looking somehow in reverse direction which means that one wants to replaces each bond coupling two spins by a set of spins and bonds coupling the two spins, for MKRG in the particular form that each bond is replaced by b parallel branches consisting each of s bonds, see Fig. 1a. This process is repeated G times, such that, if one starts from one bond, a hierarchical lattice is formed. This means, after G generation, the distance between the first and the last spin is $L = s^G$ and one has $N = (bs)^G$ bonds. Defining a fractal dimension d_f via $N = L^{d_f}$, i.e. $d_f = \ln N / \ln L$,

one obtaines $d_f = 1 + \ln b / \ln s$ as effective dimension of the system. In particular $d = 2$, if $s = b$ as in Ref. [1].

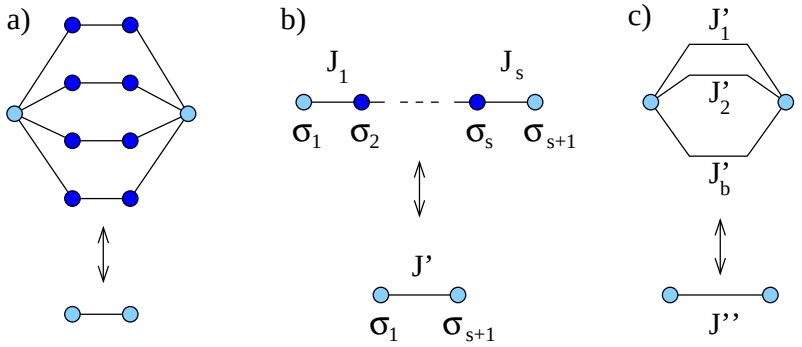

Figure 1. Migdal-Kadanoff transformation. a) b branches of each s bonds are merged into one bond by decimating the spins shown in dark. b) Each branch can be considered as a one-dimensional Ising model, where the spins except the first and the last are decimated. c) The final bond is just the sum of the bonds of the branches.

To understand how the functional transformation works, one first considers a chain with bonds $J_1, J_2, \ldots J_s$ with arbitrary values, i.e., drawn from the underlying distribution of bonds, $H = -\sum_{i=1}^{s} J_{i,i+1}\sigma_i\sigma_{i+1}$ ($\sigma_i = \pm 1$). By using the transformation $\sigma_{i+1} = \sigma_i\sigma'_{i+1}$, leading to $\sigma_i\sigma_{i+1} = \sigma_i\sigma_i\sigma'_{i+1} = \sigma'_{i+1}$, the partition function at inverse temperature β is readily calculated since it factorizes :

$$Z = \sum_{\{\sigma\}} \exp(-\beta H) = \sum_{\{\sigma_1 \ldots \sigma_{s+1}\}} \prod_{i=1}^{s} \exp\left(\beta J_{i,i+1}\sigma_i\sigma_{i+1}\right)$$

$$= \sum_{\sigma_1} \prod_{i=1}^{s} \left(\sum_{\{\sigma'_{i+1}\}} \exp\left(\beta J_{i,i+1}\sigma'_{i+1}\right) \right) = 2 \prod_{i=1}^{s} \cosh\left(\beta J_{i,i+1}\right)$$

Using the same transformation, via $\sigma_1\sigma_{s+1} = \sigma_1\sigma_s\sigma'_{s+1} = \ldots = \sigma_1^2\sigma'_2 \ldots \sigma'_{s+1}$:

$$\langle \sigma_1\sigma_{s+1} \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \sigma_1\sigma_{s+1} \exp(-\beta H)$$

$$= \frac{2}{Z} \prod_{i=1}^{s} \sinh\left(\beta J_{i,i+1}\right) = \prod_{i=1}^{s} \tanh\left(\beta J_{i,i+1}\right) .$$

Since after decimation of spins $\sigma_2 \ldots \sigma_s$, leading to the two spin system with bond J', the same partition function and in particular the same correlation must be obtained, this results in $\tanh(\beta J') = \prod_{i=1}^{s} \tanh\left(\beta J_{i,i+1}\right)$. Finally, the bond of the full system is just the sum of the resulting bonds of the b branches, leading to Eq. (2) of Ref. [1]. Numerically, one treats G generations of the transformation using a "population dynamics" approach [12]. This means, in principle, one initializes a large set of K bonds drawn from the original distribution and draws new generations each time by K times taking a number of sb bonds from the current generation $G-1$ to calculate one member of the next generation G.

The RG approach, albeit not being exact, has the advantages that the approach is very intuitive and large effective system sizes can often be studied easily, such that effects stemming from finite-size limitations can be traced down easily. This allows to get the right perspective on results of, e.g., Monte Carlo simulations, where system-size limitations are usually much harder to overcome. One example is the three-dimensional spin glass, where a MKRG study showed [13] that closely below the phase transition point even the simple Migdal-Kadanoff spin glass *appears* to have a complicated behavior, corresponding to "replica-symmetry breaking" [14], if only small systems are studied. For large systems, or sufficiently far below the phase transition temperature, one observes the trivial two-state "droplet" behavior [15, 16] which is intrinsic to the MK spin glass. This does not mean that the low-temperature behavior of real 3d spin glasses is simple, it only shows what precautions one has to take to be not mislead by finite-size effects of numerical simulations.

Also for the case of two-dimensional spin glasses, where $T_c = 0$ and the model is paramagnetic for $T > 0$, such finite-size effects had for quite a while hindered the correct understanding of the low-temperature $T \to 0$ behavior. Earlier results indicated that systems with Gaussian and with bimodal behavior behave differently. This was most prominently visible by studying the divergence of the correlation length ξ when approaching the phase transition temperature $T_c = 0$. For the Gaussian model a power-law divergence $\xi \sim T^{-\nu}$ was observed [17], while for the bimodal $\pm$ model in different numerical studies both power-law [18, 19, 20] and apparent exponential divergence [21],[22]‡ $\xi \sim e^{2J/T}$ were discussed. An exponential divergence for the bimodal spin glass seemed in accordance with the fact that for the Gaussian model the $T = 0$ stiffness exponent $\theta_{\rm G} \approx -0.287$ while $\theta_{\pm} = 0$ for the bimodal case [23, 24]. The stiffness exponent θ describes how the ground-state energy difference ΔE for systems with periodic and anti-periodic boundary conditions evolves with growing system sizes. Hence, for $\theta \leq 0$, no stable long-range order can exist. The stiffness exponent is for two-dimensional spin glasses, where $T_c = 0$, related through, again, a simple RG argument [25]§ to the correlation-length exponent via $-1/\nu = \theta$. Hence, setting $\theta = 0$ as for the bimodal case, leads directly to a divergence faster than any power, i.e., exponential.

Nevertheless, more recent Monte Carlo studies [26] contained results that indicated that through sophisticated finite-size scaling analysis, and also by studying the temperature-behavior of the correlation length, both Gaussian and bimodal models might show universal behavior at small but nonzero temperature, if only the system size is large enough, i.e., above some temperature-dependent crossover length. This was confirmed by a couple of subsequent studies [27, 28]. Nevertheless, the system size accessible via Monte Carlo simulations is still limited (although via a recetly developed partition-function-based approach of Ref. [28] rather large systems of $L = 512$ are feasible). Also the interpretation of the result is somehow indirectly since the data analysis involves usually extensive use of finite-size data collapse plots. A somehow more intuitive, but still incomplete understanding could be obtained by studying droplet excitations which dominate the low-temperature behavior and by using exact ground-state algorithms where large system sizes are feasible. Here, it turned out that these droplets indeed behave for large system sizes the same for Gaussian and bimodal spin glass [29, 30, 31], as described by the same or similar droplets exponents

‡ See also *Papercore* summary `http://www.papercore.org/Houdayer2001`.
§ See also *Papercore* summary `http://www.papercore.org/Bray1984`.

$\theta_{\pm}^{\mathrm{dp}} \approx \theta_{\mathrm{G}}^{\mathrm{dp}} = \theta_{\mathrm{G}}$. This offers maybe an explaining why for large large systems, even without the need to refer to a temperature-dependent second crossover length l_c, Gaussian and bimodal model might behave the same at finite temperatures.

Now, in the work of Jörg and Krzakala [1], the MKRG approach allowed for a much better and very descriptive understanding of what is going on. Here, the stiffness can be read of from the second moment of the interactions via $\langle J'^2 \rangle \sim L^{\theta} = s^{G\theta}$. One can actually "see" how the behavior changes when going to larger and larger systems sizes, corresponding to an increasing number G of MKRG iterations (but only up to order of 50 iterations where necessary). For example when looking at the behavior of the stiffness exponent, one observes for both models paramagnetic behavior, corresponding to $\theta = -\infty$, while below a crossover length scale ξ_{eq}, both systems are governed by the $T = 0$ exponent θ_{G}, which takes the value about -0.278 in the MKRG approximation. For the bimodal system, there exist a second crossover length $l_{\mathrm{c}}(T) < \xi_{\mathrm{eq}}(T)$, below which the stiffness exponent is zero, thus explaining the apparently different behavior observed in earlier studies.

To summarize, the MKRG approach, although the actual values for the critical exponents (or critical values for other models) might be not fully precise, is still a very versatile tool when it comes to get an *understanding* of the origin of many observed behaviors, and it will still be certainly useful in the future for many other studies. Recent applications include, to name only a few, the Heisenberg model [32], percolation [33], random Potts models [34], or polymers [35]. Furthermore, for the spin-glass case, it would be interesting to study droplet excitations via MKRG (if possible), since they seem to behave similar at exact zero temperature for both types of systems (at large scales). Droplets are probably more relevant for the thermodynamic behavior in contrast to domain-wall excitations, where the crossover length l_c diverges for $T \to 0$ such that at exact $T = 0$ Gaussian and bimodal model are always different.

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