

# Diffusion with stochastic resetting on a lattice

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We provide an exact formula for the mean first-passage time (MFPT) to a target at the origin for a single particle diffusing on a  $d$ -dimensional hypercubic *lattice* starting from a fixed initial position  $\vec{R}_0$  and resetting to  $\vec{R}_0$  with a rate  $r$ . Previously known results in the continuous space are recovered in the scaling limit  $r \rightarrow 0$ ,  $R_0 = |\vec{R}_0| \rightarrow \infty$  with the product  $\sqrt{r} R_0$  fixed. However, our formula is valid for any  $r$  and any  $\vec{R}_0$  that enables us to explore a much wider region of the parameter space that are inaccessible in the continuum limit. For example, we have shown that the MFPT, as a function of  $r$  for fixed  $\vec{R}_0$ , diverges in the two opposite limits  $r \rightarrow 0$  and  $r \rightarrow \infty$  with a unique minimum in between, provided the starting point is not a nearest neighbour of the target. In this case, the MFPT diverges as a power law  $\sim r^\phi$  as  $r \rightarrow \infty$ , but very interestingly with an exponent  $\phi = (|m_1| + |m_2| + \dots + |m_d|) - 1$  that depends on the starting point  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$  where  $a$  is the lattice spacing and  $m_i$ 's are integers. If, on the other hand, the starting point happens to be a nearest neighbour of the target, then the MFPT decreases monotonically with increasing  $r$ , approaching a universal limiting value 1 as  $r \rightarrow \infty$ , indicating that the optimal resetting rate in this case is infinity. We provide a simple physical reason and a simple Markov-chain explanation behind this somewhat unexpected universal result. These interesting results on a lattice are not captured by the continuum theory. Our analytical predictions are verified in numerical simulations on lattices up to 50 dimensions. Finally, in the absence of a target, we also compute exactly the position distribution of the walker in the nonequilibrium stationary state that also displays interesting lattice effects not captured by the continuum theory.

## I. INTRODUCTION

Stochastic resetting has emerged as an active area of research in statistical physics over the past decade, finding numerous applications across diverse fields from stochastic processes to random search algorithms. Stochastic resetting simply means interrupting the natural dynamics of a process (classical or quantum) at random times and restarting the process. Perhaps the simplest model of stochastic resetting corresponds to a single particle diffusing in continuous space starting from a fixed initial position and resetting to this initial position at a constant rate  $r$  [1]. The exact solution of this model had two interesting predictions [1, 2]: (i) the mean first-passage time (MFPT) to find a fixed target located at the origin is finite and as a function of  $r$ , displays a unique minimum at  $r = r^*$  that depends on the initial distance from the target and (ii) in the absence of a target, the resetting at a constant rate  $r$  drives the system into a nonequilibrium stationary state at long times with a non-Gaussian position distribution. Following (i), one can then set the resetting rate at the minimum value  $r^*$  to minimize the MFPT and thus expedite the search of the target. The existence of a finite optimal resetting rate  $r^*$  has rendered the stochastic resetting as an efficient mechanism to speed up a search process. This model subsequently triggered a flurry of activities where both aspects of resetting (i) and (ii) were investigated, in theoretical models as well as in experiments (for reviews see, [3–6]). For example, in one dimension, the MFPT of a diffusing particle starting at the initial position  $x_0$  to

a target located at the origin was found to have a very simple expression [1]

$$\langle T \rangle_r(x_0) = \frac{1}{r} \left[ e^{\sqrt{\frac{r}{D}} x_0} - 1 \right]. \quad (1)$$

The MFPT diverges in the two limits  $r \rightarrow 0$  and  $r \rightarrow \infty$  and has a minimum at some intermediate  $r^*$ . The divergence as  $r \rightarrow 0$  issues from the fact that in the absence of resetting the ‘bad’ trajectories that take the walker away from the origin occur with high probability and contribute to the mean capture time making it infinite. In the opposite limit, when there are many resettings, the trajectory essentially gets localised at its starting position and the walker fails to reach the target making the MFPT divergent. This result was generalised to higher dimensions [7], where the target needs to have a finite size in  $d > 2$  to be captured with a nonzero probability by the random walker modelled as a point particle. These results for the MFPT in  $d = 1$  and  $d = 2$  were verified in experiments on colloidal particles in optical traps [8–10].

While the MFPT  $\langle T \rangle_r(\vec{R}_0)$  of this single diffusing particle, starting at the initial position  $\vec{R}_0$  in  $d$  dimensions and with a constant resetting rate  $r$ , is well understood when the diffusion takes place in continuous space, a natural question is to wonder what happens to the MFPT for diffusion on a lattice in  $d$  dimensions. For example, does the MFPT on a lattice, as a function of  $r$  and  $\vec{R}_0$ , exhibit more or less similar behavior as in the continuous space, or does it have new interesting regimes? The lattice computation is considerably more difficult than in the continuum, so there is no point in just repeating the

computations if they yield qualitatively similar results. But if there are new regimes as a function of  $r$  and  $\vec{R}_0$  in the lattice problem that is not captured by the continuous space result, it would indeed be interesting and worth pursuing the lattice calculations. In fact, the purpose of this paper is to show, via exact calculation of this MFPT on a  $d$ -dimensional hypercubic lattice, verified by accompanying numerical simulations and a simple Markov-chain representation, that there are very interesting behaviors displayed by lattice MFPT that are not captured by its continuous space counterpart.

Some stochastic processes in the presence of resetting have been studied on lattices [11–17] and networks [18]. In Ref. [19], a resetting random walk model on a  $d$ -dimensional lattice in discrete time was studied, but the main objective there was to compute the mean number of distinct sites visited by such a resetting random walker of  $N$  steps. However, we have not come across any explicit result for the MFPT for a standard random walker on a  $d$ -dimensional lattice, evolving in continuous time with a nonzero resetting rate  $r$ . Our goal here is to present an exact formula for the MFPT of a lattice random walker with resetting, valid in arbitrary dimension. We will see that the results from the lattice model are more general and includes interesting new regimes with strikingly unexpected behavior of the MFPT, in particular for large resetting rate. These results are completely missed by the continuous space model.

Let us briefly summarize our main results and also lay out the organization of the paper. In Section II, we first define our model precisely on a lattice and derive a general relation between the MFPT in the presence of resetting and the free propagator of the underlying process without resetting. This relation, based on renewal arguments, turns out to be very general and holds for arbitrary Markov processes, both on the lattice as well as in the continuum. This exact relation is given in Eq. (19). In the next section III, we evaluate this exact formula for a  $d$ -dimensional hypercubic lattice leading to our main explicit formula in Eq. (33). We show how to recover the continuous space results from this lattice MFPT formula, in the limit when  $r \rightarrow 0$ ,  $R_0 \rightarrow \infty$  but keeping  $\sqrt{r} R_0$  fixed.

In Section IV we describe the event-driven algorithm we have used to verify our analytical results numerically and to obtain results on other properties of the walk, in particular the number of steps taken by the walker to find the target.

In Section V, we consider several special cases of our main formula for lattice MFPT in Eq. (33) and derive explicit results. We start with  $d = 1$  and  $d = 2$  and then present some exact results in general dimensions in the two limits  $r \rightarrow 0$  and  $r \rightarrow \infty$  for a fixed starting position  $\vec{R}_0$ . We denote the starting position  $\vec{R}_0 = a \vec{m}$  with  $\vec{m} \equiv (m_1, m_2, \dots, m_d)$  where  $m_i$ 's are integers and  $a$  is the lattice constant. We show that as long as the starting position is not a nearest neighbour of the target at the origin, the MFPT first decreases as a function of

increasing  $r$ , achieves a global minimum at some  $r = r^*$  and then grows again and finally diverges as a power law as  $r \rightarrow \infty$

$$\langle T \rangle_r(\vec{R}_0) \approx \frac{\prod_{i=1}^d \Gamma(|m_i| + 1)}{\Gamma\left(\sum_{i=1}^d |m_i| + 1\right)} r^\phi \quad (2)$$

with  $\phi = (|m_1| + |m_2| + \dots + |m_d|) - 1$ . Thus the exponent  $\phi$  depends on the starting distance. This interesting new regime is inaccessible in the continuum theory where one has already taken the  $r \rightarrow 0$  limit. Another interesting explicit and general result is the following: we show that if the walker instead starts from a lattice site which is an immediate neighbour of the target, then the MFPT decreases monotonically with increasing  $r$ , achieving its minimum value 1 (universal in all dimensions) as  $r \rightarrow \infty$ . Thus in this case, the optimal resetting rate  $r^*$  is actually infinite, while it is finite if the starting site is not an immediate neighbour of the target! Hence, rather remarkably, for a starting site just next to the target, one needs to implement an infinite resetting rate to minimize the MFPT! This is another strikingly surprising result that is completely missed by the continuum limit. We also compare our analytical predictions with numerical simulations up to 50 dimensions, finding excellent agreement. For some of our results we also provide a simple explanation based on the properties of a few-state Markov process.

In the absence of a target, we show in Section VI that the system is driven at long times into a nonequilibrium stationary state (NESS). We compute the exact position distribution in the NESS for a resetting walker on a  $d$ -dimensional hypercubic lattice. Finally we conclude in Section VII and relegate some details of calculations in Appendix A.

## II. MEAN FIRST-PASSAGE TIME TO A TARGET FOR A RESETTING RANDOM WALKER ON A LATTICE: A GENERAL FORMULA

We consider a resetting random walker on an infinite  $d$ -dimensional hypercubic lattice with lattice constant  $a$ . The position of the walker evolves in continuous time, starting at the initial position  $\vec{R}_0$  at time  $t = 0$ . Let  $\vec{R}$  be the current location of the walker at time  $t$ . Then in a small time  $dt$ , the walker performs the following stochastic movements:

- it hops to any one of the  $2d$  neighbouring sites  $\vec{R} + a \vec{e}$  of  $\vec{R}$  with probability  $dt$ , i.e. with rate  $\lambda_{\text{hop}} = 1$
- it hops to the starting position  $\vec{R}_0$  with probability  $r dt$ , where  $r$  denotes the resetting rate

- with the remaining probability  $1 - (r + 2d) dt$ , it stays at the current location  $\vec{R}$ .

Let  $P_r(\vec{R}, \vec{R}_0, t)$  denote the probability that the walker is located at site  $\vec{R}$  at time  $t$ , starting from  $\vec{R}_0$  at  $t = 0$ .

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$$\frac{\partial P_r(\vec{R}, \vec{R}_0, t)}{\partial t} = \left[ \sum_{\vec{e}} P_r(\vec{R} + a \vec{e}, \vec{R}_0, t) - 2d P_r(\vec{R}, \vec{R}_0, t) \right] - r P_r(\vec{R}, \vec{R}_0, t) + r \delta_{\vec{R}, \vec{R}_0}, \quad (3)$$


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starting from the initial condition  $P_r(\vec{R}, \vec{R}_0, 0) = \delta_{\vec{R}, \vec{R}_0}$  (Kronecker delta). The first term inside the parenthesis  $[\dots]$  on the right hand side (rhs) of Eq. (3) is just a lattice Laplacian that describes the diffusion, the second term describes the loss of probability from  $\vec{R}$  due to resetting to  $\vec{R}_0$  with rate  $r$ , while the third term describes the gain in probability at  $\vec{R}_0$  via resetting from other sites. We will later present an exact solution of the Fokker-Planck equation (3) in Section VI.

We note that Eq. (3) is just the lattice version of the Fokker-Planck equation in continuous space introduced and studied in Refs. [1, 2]. To see how the lattice Fokker-Planck equation (3) reduces to its continuous counterpart, we take the continuum limit  $a \rightarrow 0$ , where  $a$  is the lattice length. Expanding the right hand side (rhs) of Eq. (3) up to quadratic order in  $a$ , one gets

$$\frac{\partial P_r(\vec{R}, \vec{R}_0, t)}{\partial t} \approx a^2 \nabla_{\vec{R}}^2 P(\vec{R}, \vec{R}_0, t) - r P_r(\vec{R}, \vec{R}_0, t) + r \delta_{\vec{R}, \vec{R}_0}, \quad (4)$$

Next we need to rescale time  $t = \tilde{t}/a^2$  and consequently the resetting rate  $r = \tilde{r} a^2$ . Furthermore, the probability density  $\tilde{P}$  in continuous space is related to the probability  $P$  on a lattice via the relation  $P = \tilde{P} a$ . Under this rescaling, Eq. (4) reduces to the standard continuous Fokker-Planck equation studied in Refs. [1, 2] (upon setting the diffusion constant  $D = 1$ )

$$\frac{\partial \tilde{P}(\vec{R}, \vec{R}_0, t)}{\partial \tilde{t}} = \nabla_{\vec{R}}^2 \tilde{P}(\vec{R}, \vec{R}_0, t) - \tilde{r} \tilde{P}(\vec{R}, \vec{R}_0, t) + \tilde{r} \delta(\vec{R} - \vec{R}_0), \quad (5)$$

where we replaced  $\delta_{\vec{R}, \vec{R}_0}/a$  by the Dirac delta function in the limit  $a \rightarrow 0$ . Thus to recover the continuum limit, we need to implement the following rescaling for the time and the resetting rate

$$t = \frac{\tilde{t}}{a^2} \quad \text{and} \quad r = \tilde{r} a^2, \quad (6)$$

and take the limit  $a \rightarrow 0$ . Hence one not only needs to take the large distance (in units of  $a$ ) and late time

The subscript  $r$  in  $P_r$  denotes the presence of resetting move with rate  $r$ . This probability evolves via the Fokker-Planck equation

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limits, but needs additionally to take the vanishing reset rate  $r \rightarrow 0$  limit in an appropriate way.

We now assume that there is a target at the origin  $\vec{0}$  and we are interested in computing the MFPT  $\langle T \rangle_r(\vec{R}_0)$  to the target, starting at  $\vec{R}_0 = a \vec{m}$ , where  $\vec{m}$  is a lattice site  $(m_1, m_2, \dots, m_d)$  with  $m_i$ 's integers. Let  $F_r(\vec{R}_0, t)$  denote the first-passage probability density to the target, i.e., the probability density to reach the target for the first time at time  $t$ , starting at  $\vec{R}_0$ . Let  $Q_r(\vec{R}_0, t)$  denote the survival probability of the walker up to time  $t$ , i.e., the probability that it does not reach the target up to time  $t$ . Then clearly it is related to the first-passage probability density via the simple relation [20, 21]

$$Q_r(\vec{R}_0, t) = \int_t^\infty F(\vec{R}_0, t') dt'$$

implying

$$F_r(\vec{R}_0, t) = -\frac{\partial Q_r(\vec{R}_0, t)}{\partial t}. \quad (7)$$

Consequently the MFPT can be expressed in terms of the survival probability

$$\langle T \rangle_r(\vec{R}_0) = \int_0^\infty t F_r(\vec{R}_0, t) dt = \int_0^\infty Q_r(\vec{R}_0, t) dt, \quad (8)$$

where we used the relation in Eq. (7) and performed an integration by parts (assuming  $Q_r(\vec{R}_0, t)$  decays faster than  $1/t$  as  $t \rightarrow \infty$ ). Let us also define, for future usage, the Laplace transform of the survival probability

$$\tilde{Q}_r(\vec{R}_0, s) = \int_0^\infty Q_r(\vec{R}_0, t) e^{-st} dt. \quad (9)$$

Thus, the result (8) then reduces to

$$\langle T \rangle_r(\vec{R}_0) = \tilde{Q}_r(\vec{R}_0, s = 0). \quad (10)$$

Now, the survival probability  $Q_r(\vec{R}_0, t)$  in the presence of resetting can be related to the survival probability  $Q_0(\vec{R}_0, t)$  in the absence of resetting, by using a renewal approach [1, 3, 22, 23]. The renewal approach leads to the exact equation

$$Q_r(\vec{R}_0, t) = e^{-rt} Q_0(\vec{R}_0, t) + r \int_0^t d\tau e^{-r\tau} Q_0(\vec{R}_0, \tau) Q_r(\vec{R}_0, t - \tau) \quad (11)$$

It is easy to understand the two terms on the rhs of Eq. (12). The first term corresponds to the event when there is no resetting in  $[0, t]$  which happens with probability  $e^{-rt}$ . In this case, the survival probability is simply  $Q_0(\vec{R}_0, t)$  which, multiplied by the factor  $e^{-rt}$ , gives the first term. The second term corresponds to the cases when there is one or more resetting events in  $[0, t]$ . In this case, let  $\tau$  denote the time at which the first resetting occurs. The probability for the first resetting to occur in  $[\tau, \tau + d\tau]$  is simply  $r e^{-r\tau} d\tau$ . Prior to this resetting the target survives with probability  $Q_0(\vec{R}_0, \tau)$  and after the first resetting, the process renews and one needs to multiply by  $Q_r(\vec{R}_0, t - \tau)$  to ensure survival during the rest of the interval of duration  $t - \tau$ . Finally the first resetting epoch  $\tau$  can occur anywhere in  $[0, t]$  and one needs to integrate over  $\tau \in [0, t]$  to obtain the second term in (12). The convolution structure of the second term in (12) immediately suggests to take a Laplace transform of this equation with respect to  $t$ . Performing this Laplace transform, one gets a very simple general relation

$$\tilde{Q}_r(\vec{R}_0, s) = \frac{\tilde{Q}_0(\vec{R}_0, s + r)}{1 - r \tilde{Q}_0(\vec{R}_0, s + r)}. \quad (12)$$

Let us note that there is a shift in the Laplace variable from  $s$  to  $s + r$  on the rhs of (12). Consequently setting  $s = 0$  and using (10), the MFPT in the presence of resetting can be expressed explicitly in terms of the Laplace transform of the survival probability in the absence of resetting

$$\langle T \rangle_r(\vec{R}_0) = \frac{\tilde{Q}_0(\vec{R}_0, r)}{1 - r \tilde{Q}_0(\vec{R}_0, r)}. \quad (13)$$

It is convenient to express the rhs of Eq. (13) in terms of the Laplace transform of the first-passage probability density (without resetting). This can be done by recalling that the first-passage probability density is related to the survival probability via the relation

$$F_0(\vec{R}_0, t) = -\partial_t Q_0(\vec{R}_0, t). \quad (14)$$

Taking Laplace transform with respect to  $t$  yields

$$\tilde{F}_0(\vec{R}_0, s) = 1 - s \tilde{Q}_0(\vec{R}_0, s). \quad (15)$$

Using this relation in Eq. (13), the MFPT can then be expressed as

$$\langle T \rangle_r(\vec{R}_0) = \frac{1}{r} \left[ \frac{1}{\tilde{F}_0(\vec{R}_0, r)} - 1 \right]. \quad (16)$$

Thus, thanks to this explicit relation, computing the MFPT in the presence of resetting just requires the knowledge of the Laplace transform of the first-passage probability density in the absence of resetting.

We now show how to express this Laplace transform  $\tilde{F}_0(\vec{R}_0, s)$  for a pure (without resetting) random walker

on a  $d$ -dimensional lattice in terms of the lattice Green's function. To find this relation, consider a random walk trajectory that starts at  $\vec{R}_0$  and arrives at the origin at time  $t$ . The probability for this event is simply  $P_0(\vec{0}, \vec{R}_0, t)$ . Now, this path that arrives at the origin at time  $t$  must have hit the origin for the first time at some epoch before  $t$ , say at  $\tau$  and then has returned to the origin at time  $t$  (possibly  $t = \tau$  if it is the first visit). Hence one can again use a renewal equation to connect the first-passage probability density and the return probability

$$P_0(\vec{0}, \vec{R}_0, t) = \int_0^t d\tau F_0(\vec{R}_0, \tau) P_0(\vec{0}, \vec{0}, t - \tau). \quad (17)$$

Taking Laplace transform with respect to time gives

$$\tilde{F}_0(\vec{R}_0, s) = \frac{\tilde{P}_0(\vec{0}, \vec{R}_0, s)}{\tilde{P}_0(\vec{0}, \vec{0}, s)} \quad \text{for } \vec{R}_0 \neq \vec{0}. \quad (18)$$

The Laplace transform  $\tilde{P}_0(\vec{R}, \vec{R}_0, s)$  (or its analogue generating function when the walk takes place in discrete time steps) is usually referred to as the lattice Green's function (for a nice review on lattice Green's function, see Ref. [24]). The ratio of these lattice Green's function also appears in the computation of the mean number of distinct sites visited by a random walker on a lattice [25, 26], see also the recent Ref. [19] in the context of a resetting random walker.

Substituting the result (18) in Eq. (16) then expresses the MFPT with resetting in terms of the lattice Green's function without resetting

$$\langle T \rangle_r(\vec{R}_0) = \frac{1}{r} \left[ \frac{\tilde{P}_0(\vec{0}, \vec{0}, r)}{\tilde{P}_0(\vec{0}, \vec{R}_0, r)} - 1 \right]. \quad (19)$$

We note that this result (19) is actually very general, and holds even for random walks in the presence of a force or potential. It is valid in all dimensions. It is also valid for a Brownian motion in  $d$ -dimensions in the presence or absence of a drift, where  $\tilde{P}_0(\vec{R}, \vec{R}_0, s)$  is just the Laplace transform of the propagator of the process. For example, for a Brownian motion in one dimension, the propagator in real time is simply

$$P_0(R, R_0, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(R-R_0)^2/4Dt}. \quad (20)$$

Its Laplace transform is simply

$$\begin{aligned} \tilde{P}_0(\vec{R}, \vec{R}_0, s) &= \int_0^\infty dt e^{-st} \frac{1}{\sqrt{4\pi Dt}} e^{-(R-R_0)^2/4Dt} \\ &= \frac{1}{\sqrt{4sD}} e^{-\sqrt{s/D}|R-R_0|}. \end{aligned} \quad (21)$$

Substituting (21) in (19), one recovers the continuous space result in Eq. (1).

The expression in Eq. (19) is our main result which says that to compute the MFPT in the presence of resetting, we just need to know the Laplace transform of

the propagator of the underlying process without resetting. We show in the next subsection how to evaluate it explicitly for a random walk on a  $d$ -dimensional hypercubic lattice.

### III. AN EXPLICIT FORMULA FOR THE MFPT FOR A RESETTING $d$ -DIMENSIONAL RANDOM WALK ON A HYPERCUBIC LATTICE

Consider a reset free ( $r = 0$ ) random walk, whose position distribution evolves via the lattice diffusion equation

$$\frac{\partial P_0(\vec{R}, \vec{R}_0, t)}{\partial t} = \left[ \sum_{\vec{e}} P_0(\vec{R} + a\vec{e}, \vec{R}_0, t) - 2d P_0(\vec{R}, \vec{R}_0, t) \right], \quad (22)$$

obtained by setting  $r = 0$  in Eq. (3). It starts from the initial condition  $P_0(\vec{R}, \vec{R}_0, t = 0) = \delta_{\vec{R}, \vec{R}_0}$ . This linear equation can be solved exactly using the Laplace-Fourier transform [26]. Let us first take the Laplace transform of (22) with respect to  $t$ , by defining

$$\tilde{P}_0(\vec{R}, \vec{R}_0, s) = \int_0^\infty P_0(\vec{R}, \vec{R}_0, t) e^{-st} dt. \quad (23)$$

Using the initial condition, we get

$$(s + 2d) \tilde{P}_0(\vec{R}, \vec{R}_0, s) - \delta_{\vec{R}, \vec{R}_0} = \sum_{\vec{e}} \tilde{P}_0(\vec{R} + a\vec{e}, \vec{R}_0, s). \quad (24)$$

Next we define the Fourier transform

$$\hat{P}_0(\vec{k}, s) = \sum_{\vec{R}} \tilde{P}_0(\vec{R}, \vec{R}_0, s) e^{i\vec{k} \cdot \frac{(\vec{R} - \vec{R}_0)}{a}}. \quad (25)$$

Taking Fourier transform of (24) gives

$$\hat{P}_0(\vec{k}, s) = \frac{1}{\left[ (s + 2d) - 2 \sum_{i=1}^d \cos(k_i) \right]}. \quad (26)$$

Finally, inverting the Fourier transform and using symmetries, one gets the well known expression for the lattice Green's function [24, 26]

$$\tilde{P}_0(\vec{R}, \vec{R}_0, s) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dk_d}{2\pi} \frac{e^{-i\vec{k} \cdot \frac{(\vec{R} - \vec{R}_0)}{a}}}{\left[ (s + 2d) - 2 \sum_{i=1}^d \cos(k_i) \right]}. \quad (27)$$

We next set  $\vec{R} = \vec{0}$  and  $\vec{R}_0 = a\vec{m}$  where  $\vec{m} \equiv (m_1, m_2, \dots, m_d)$  with  $m_i$ 's being integers. Using the

symmetry in the  $k$  space, we can then express Eq. (27) as a  $d$ -dimensional real integral

$$\tilde{P}_0(\vec{0}, \vec{R}_0 = a\vec{m}, s) = \int_{-\pi}^{\pi} \frac{dk_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{dk_d}{2\pi} \frac{\prod_{i=1}^d \cos(k_i m_i)}{\left[ (s + 2d) - 2 \sum_{i=1}^d \cos(k_i) \right]}. \quad (28)$$

To make further progress, we use the integral representation

$$\frac{1}{z} = \int_0^\infty dt e^{-tz}, \quad (29)$$

to rewrite Eq. (28) as

$$\tilde{P}_0(\vec{0}, \vec{R}_0 = a\vec{m}, s) = \frac{1}{(s + 2d)} \times \int_0^\infty dt e^{-t} \prod_{i=1}^d \int_{-\pi}^{\pi} \frac{dk_i}{2\pi} e^{-\frac{2t}{(s+2d)} \cos(k_i)} \cos(k_i m_i). \quad (30)$$

Next we use the identity [27]

$$\int_{-\pi}^{\pi} \frac{dk}{2\pi} \exp[z \cos(k)] \cos(km) = I_{|m|}(z), \quad (31)$$

where  $I_m(z)$  is the modified Bessel function of the first kind with index  $m$  and argument  $z$ . This identity (31) is valid only when  $m$  is an integer, as in our case. Using (31) in (30) we get a compact formula

$$\tilde{P}_0(\vec{0}, \vec{R}_0 = a\vec{m}, s) = \frac{1}{(s + 2d)} \int_0^\infty dt e^{-t} \prod_{i=1}^d I_{|m_i|} \left( \frac{2t}{(s + 2d)} \right). \quad (32)$$

Substituting this result in Eq. (19), we get an explicit expression for the MFPT of the lattice walker with resetting rate  $r$  and starting from  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$

$$\langle T \rangle_r (\vec{R}_0 = a\vec{m}) = \frac{1}{r} \left[ \frac{\int_0^\infty dt e^{-t} \left[ I_0 \left( \frac{2t}{(r+2d)} \right) \right]^d}{\int_0^\infty dt e^{-t} \prod_{i=1}^d I_{|m_i|} \left( \frac{2t}{(r+2d)} \right)} - 1 \right]. \quad (33)$$

This is our main new general result valid in arbitrary dimension and arbitrary starting point  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$ . Unfortunately, the integrals on the rhs of (33) can not be computed in closed form for general dimension  $d$  and general  $\vec{R}_0$ . However, the representation in (33) is still nice because it can be easily evaluated numerically in Mathematica, for any  $d$  and any choice of  $\vec{R}_0$ . We analyse later (33) in several special cases where the integrals can be done exactly to make the formula for the MFPT even more explicit.

### A. Recovering the continuous space limit results

To recover the continuous space results from our exact lattice MFPT in Eq. (33), we need to take  $a \rightarrow 0$  limit and also rescale the resetting rate  $r = a^2 \tilde{r}$  and define the rescaled MFPT  $\tilde{T} = a^2 T$  (see Eq. (6)). In Appendix A we show in detail how to take this limit in Eq. (33) to obtain, in arbitrary dimension  $d$ ,

$$\langle \tilde{T} \rangle_{\tilde{r}}(\vec{R}_0) = \lim_{a \rightarrow 0} \langle T \rangle_{r=a^2 \tilde{r}}(\vec{R}_0) = \frac{1}{\tilde{r}} \left[ \frac{\Gamma(|\nu|) |\nu|^{-1} (\tilde{r})^{-|\nu|/2} a^{\nu-|\nu|}}{R_0^\nu K_\nu(R_0 \sqrt{\tilde{r}})} - 1 \right] \quad (34)$$

with  $\nu = 1 - \frac{d}{2}$  and where  $K_\nu(z)$  is the modified Bessel function of the second kind with index  $\nu$  and argument  $z$ . For  $d = 2$ , i.e.,  $\nu = 0$ , one obtains a slightly different behavior with a logarithmic correction. For  $d < 2$ , the limit  $a \rightarrow 0$  exists and is finite, while for  $d > 2$ , one needs to keep a small but nonzero lattice constant  $a$ . In Ref. [7], the MFPT was computed directly in the continuum limit by assuming that the target is spherical with a finite radius  $\epsilon$  and it was found that

$$\langle \tilde{T} \rangle_{\tilde{r}}(\vec{R}_0) = \frac{1}{\tilde{r}} \left[ \frac{\epsilon^\nu K_\nu(\epsilon \sqrt{\tilde{r}})}{R_0^\nu K_\nu(R_0 \sqrt{\tilde{r}})} - 1 \right]. \quad (35)$$

One can now take the limit  $\epsilon \rightarrow 0$  using the following asymptotic small  $z$  behavior of the Bessel function

$$K_\nu(z) \approx \begin{cases} \Gamma(|\nu|) 2^{|\nu|-1} z^{-|\nu|} & \text{as } z \rightarrow 0, |\nu| > 0 \\ -\ln(z/2) - \gamma_E & \text{as } z \rightarrow 0, |\nu| = 0, \end{cases} \quad (36)$$

where  $\gamma_E$  is the Euler gamma constant. Using this in Eq. (35), one indeed recovers Eq. (34) upon identifying  $\epsilon$  with the lattice constant  $a$ . Finally, let us remark again that the continuum result (34) is less richer than our lattice result in (33). The lattice MFPT result (33) has some interesting new regimes, in particular for large resetting rate  $r$  with fixed starting point  $\vec{R}_0$ , that the continuum limit misses since one needs to already take the limit  $r \rightarrow 0$  in arriving at the continuum limit.

## IV. NUMERICAL APPROACH

We implemented a simple continuous-time event-driven algorithm to simulate the behavior of the resetting random walker and to compare the numerical results to our analytical predictions. This algorithm also enables us to measure other quantities of interest, e.g., the total number of hops till the target is found and the number of hops needed to capture the target after the final reset.

The main idea of the event-based approach is to generate the times of the relevant events, hops and resets,

and to perform changes to the walker's position only at event times. Events that take place with some rate  $\lambda$  follow an exponential distribution with parameter  $\lambda$ , given by the probability density  $p(t) = \lambda e^{-\lambda t}$  ( $t > 0$ ), i.e., the probability distribution  $P(t) = \int_0^t p(t') dt' = 1 - e^{-\lambda t}$ . Actually we use  $\Delta t$  to denote the duration to the next event instead of  $t$ , which denotes the total time here. Since the distribution function can be inverted, the duration until the next event can be simply generated [28] by using the *Inversion Approach*. This means, one draws a random number  $u$  which is uniformly distributed in the interval  $[0, 1]$ . Then one applies the inverse distribution, i.e., assigns  $\Delta t = P^{-1}(u) = \log(1 - u)/\lambda$ . One writes  $\Delta t \sim \text{Exp}(\lambda)$  to indicate this generation.

Within the simulations, we assume lattice constant  $a = 1$ . We denote by  $\vec{R}$  the current position of the walker, by  $t$  the time of the last event, by  $t_r$  the time of the next reset event and by  $t_{\text{hop}}$  the time of the next hopping event. Each walker is considered until the target is found.

A hopping event will happen with rate  $2d$  since each of the possible  $2d$  hop occurs with rate  $\lambda_{\text{hop}} = 1$ . Since all hops to the neighbors have this same rate, for each occurring hop, one of the  $2d$  directions will be chosen randomly with probability  $1/(2d)$ . We also measure the total number  $n_{\text{hop}}$  of hops and the number  $n_{\text{final}}$  of hops since the last reset. The algorithm reads as follows:

```

algorithm walk( $\vec{R}_0, r, d$ )
begin
     $\vec{R} = \vec{R}_0$ ;                                \\ initialise
     $t = 0$ ;
     $n_{\text{hops}} = 0$ ;  $n_{\text{final}} = 0$ 
     $\Delta t \sim \text{Exp}(r)$ ;  $t_r = t + \Delta t$ ;        \\ first reset
     $\Delta t \sim \text{Exp}(2d)$ ;  $t_{\text{hop}} = t + \Delta t$ ;    \\ first hop
    while  $\vec{R} \neq \vec{0}$  do                          \\ while not found
    begin
        if  $t_r < t_{\text{hop}}$  then
        begin                                    \\ perform reset
             $\vec{R} = \vec{R}_0$ ;
             $n_{\text{final}} = 0$ ;
             $t = t_r$ ;
             $\Delta t \sim \text{Exp}(r)$ ;  $t_r = t + \Delta t$ ;    \\ next reset
        end
        else
        begin                                    \\ perform hop
            random direction  $\vec{d} = \pm \vec{e}_i$ ;
             $\vec{R} = \vec{R} + \vec{d}$ ;
             $n_{\text{hop}} = n_{\text{hop}} + 1$ ;
             $n_{\text{final}} = n_{\text{final}} + 1$ ;
             $t = t_{\text{hop}}$ ;
             $\Delta t \sim \text{Exp}(2d)$ ;  $t_{\text{hop}} = t + \Delta t$ ; \\ next hop
        end
    end
    return( $t, n_{\text{hops}}, n_{\text{final}}$ )
end

```

Note that all event rates are independent of the lattice

sites, so after a hop or reset, no other events need to be rescheduled.

## V. SOME EXPLICIT RESULTS

Our starting point is the exact lattice MFPT formula in Eq. (33). We will now analyse this formula explicitly in several special cases. For convenience, in the rest of the paper, we will use the following notation

$$\langle T \rangle_r(\vec{R}_0 = a \vec{m}) \equiv \langle T \rangle_r(m_1, m_2, \dots, m_d). \quad (37)$$

### A. Explicit results in $d = 1$

In  $d = 1$ , the result for the MFPT becomes fully explicit via the identity [27]

$$\int_0^\infty dt e^{-t} I_{|m|}(zt) = \frac{z^{-|m|} [1 - \sqrt{1 - z^2}]^{|m|}}{\sqrt{1 - z^2}}. \quad (38)$$

Using this identity in Eq. (33) for  $d = 1$  and simplifying, we get the explicit result

$$\langle T \rangle_r(m_1) = \frac{1}{r} \left[ \left( \frac{2}{r + 2 - \sqrt{r^2 + 4r}} \right)^{|m_1|} - 1 \right], \quad (39)$$

valid for any integer  $m_1$ . As a function of  $r$  for fixed  $|m_1|$ , it has the asymptotic behaviors

$$\langle T \rangle_r(m_1) \approx \begin{cases} \frac{|m_1|}{\sqrt{r}} & \text{as } r \rightarrow 0 \\ r^{|m_1|-1} & \text{as } r \rightarrow \infty. \end{cases} \quad (40)$$

Thus, as  $r \rightarrow 0$ , it diverges for any  $|m_1|$ . In contrast, as  $r \rightarrow \infty$ , it diverges as a power law for any  $|m_1| > 1$ , but it approaches a constant 1 for  $|m_1| = 1$ . In Fig. 1, we plot the MFPT  $\langle T \rangle_r(m_1)$  in Eq. (39) as a function of  $r$  for three different values of  $m_1 = 1, 2, 3$  (shown by solid lines) and compare them to direct numerical simulation results (symbols), finding perfect matching. For any  $|m_1| > 1$ , it then displays a unique minimum at some  $r = r^*(|m_1|)$ . However, for  $|m_1| = 1$ , it decreases monotonically to its asymptotic value 1 as  $r \rightarrow \infty$ . We will see later that this is a generic feature in any dimension. If the walker starts from a site that is nearest neighbour to the origin (target), then the MFPT decreases monotonically as  $r$  increases, approaching the universal value 1 in all dimensions as  $r \rightarrow \infty$ . However, for any starting site that is not a nearest neighbour of the origin, the MFPT displays a unique minimum at some  $r^*$  that depends on the starting site and the dimension  $d$ . We will provide later a simple physical argument for this universal asymptotic value 1 of the MFPT as  $r \rightarrow \infty$  when the starting site is a nearest neighbour of the origin.

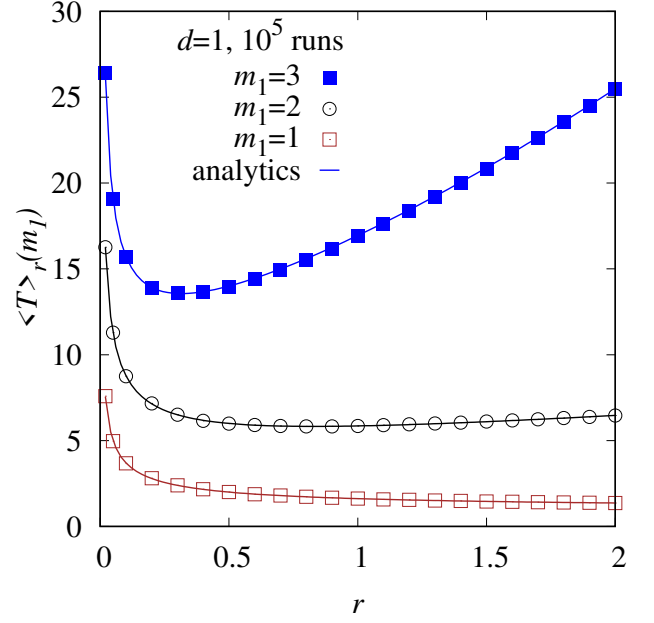


FIG. 1.  $\langle T \rangle_r(m_1)$  vs.  $r$  in one dimension ( $d = 1$ ) for  $m_1 = 1, m_1 = 2$  and  $m_1 = 3$ . While for any  $m_1 > 1$ , the curve exhibits a unique minimum at some  $r = r^*(|m_1|)$ , for  $m_1 = 1$  it decays monotonically with increasing  $r$ , approaching the limiting value 1 as  $r \rightarrow \infty$ . The solid lines correspond to the analytical formula in Eq. (39).

Finally, let us note that by setting  $m_1 = R_0/a$ ,  $T = \tilde{T} a^2$ ,  $r = \tilde{r} a^2$  in Eq. (39) and taking the  $a \rightarrow 0$  limit, we get

$$\langle \tilde{T} \rangle_{\tilde{r}}(\vec{R}_0) = \frac{1}{\tilde{r}} \left[ e^{\sqrt{\tilde{r}} R_0} - 1 \right], \quad (41)$$

thus recovering the continuous space result in Eq. (1) with diffusion constant  $D = 1$  set to unity. Thus the continuum limit corresponds to taking the limits  $r \rightarrow 0$ ,  $|m_1| \rightarrow \infty$  keeping the product  $\sqrt{r} |m_1|$  fixed in Eq. (39). Consequently, in the  $(r, m_1)$  plane, the continuum limit captures the behaviour of the MFPT only in one corner. However, the lattice result (39) holds in the full  $(r, m_1)$  plane and hence is richer. In particular, the power law growth of the MFPT  $r^{|m_1|-1}$  for large  $r$  in Eq. (40) is completely missed by the continuum limit.

### B. Explicit results in $d = 2$

In  $d = 2$ , the MFPT in Eq. (33) reads for the starting point  $\vec{R}_0 = a(m_1, m_2)$

$$\langle T \rangle_r(m_1, m_2) = \frac{1}{r} \left[ \frac{\int_0^\infty dt e^{-t} \left[ I_0 \left( \frac{2t}{(r+4)} \right) \right]^2}{\int_0^\infty dt e^{-t} I_{|m_1|} \left( \frac{2t}{(r+4)} \right) I_{|m_2|} \left( \frac{2t}{(r+4)} \right)} - 1 \right]. \quad (42)$$

The integral in the numerator on the rhs of Eq. (42) can be computed explicitly for  $|z| < 1$  [24]

$$\int_0^\infty dt e^{-t} \left[ I_0 \left( \frac{zt}{2} \right) \right]^2 = \frac{2}{\pi} K(z^2) = {}_2F_1 \left( \frac{1}{2}, \frac{1}{2}, 1, z^2 \right), \quad (43)$$

where  $K(u)$  is the EllipticK function with argument  $u$  and  ${}_2F_1$  is the standard hypergeometric function [27]. The EllipticK function is defined as

$$K(u) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - u \sin^2(\theta)}}. \quad (44)$$

Note that there was a typographical error in Ref. [24] where the argument of the hypergeometric function was reported to be  $z$ , instead of  $z^2$ . In contrast, the integral in the denominator on the rhs of Eq. (42) does not seem to be doable for general  $(m_1, m_2)$ . However, it can be done explicitly for the two cases  $(1, 0)$  and  $(1, 1)$ .

**The case  $(m_1 = 1, m_2 = 0)$ .** In this case, Eq. (42) reduces to

$$\langle T \rangle_r(1, 0) = \frac{1}{r} \left[ \frac{2}{\pi} \frac{K \left( \frac{16}{(4+r)^2} \right)}{\int_0^\infty dt e^{-t} I_0 \left( \frac{2t}{(r+4)} \right) I_1 \left( \frac{2t}{(r+4)} \right)} - 1 \right]. \quad (45)$$

To compute the integral in the denominator in (45) we first use the identity  $I_1(z) = I'_0(z)$  [27] to re-write this integral as

$$\int_0^\infty dt e^{-t} I_0(zt/2) I_1(zt/2) = \frac{1}{z} \int_0^\infty dt e^{-t} \frac{d}{dt} [I_0^2(zt/2)]. \quad (46)$$

Performing the last integral by parts and using the result from (43) we get

$$\int_0^\infty dt e^{-t} I_0(zt/2) I_1(zt/2) = \frac{1}{z} \left[ \frac{2}{\pi} K(z^2) - 1 \right], \quad (47)$$

valid again for  $|z| < 1$ . Substituting in (45) and simplifying leads to the exact result

$$\langle T \rangle_r(1, 0) = \frac{1}{r(r+4)} \left[ \frac{4}{\frac{2}{\pi} K \left( \frac{16}{(4+r)^2} \right) - 1} - r \right]. \quad (48)$$

The asymptotic behaviors are given by

$$\langle T \rangle_r(1, 0) \approx \begin{cases} -\frac{\pi}{r \ln(r)} & \text{as } r \rightarrow 0 \\ 1 + \frac{3}{r} & \text{as } r \rightarrow \infty. \end{cases} \quad (49)$$

In Fig. 2 we plot  $\langle T \rangle_r(1, 0)$  vs.  $r$  and compare with our direct numerical simulation, finding excellent agreement. We see once more that as  $r \rightarrow \infty$ , the MFPT approaches 1 monotonically given that this starting point  $(1, 0)$  is a nearest neighbour of the target at  $(0, 0)$ . Of course, by

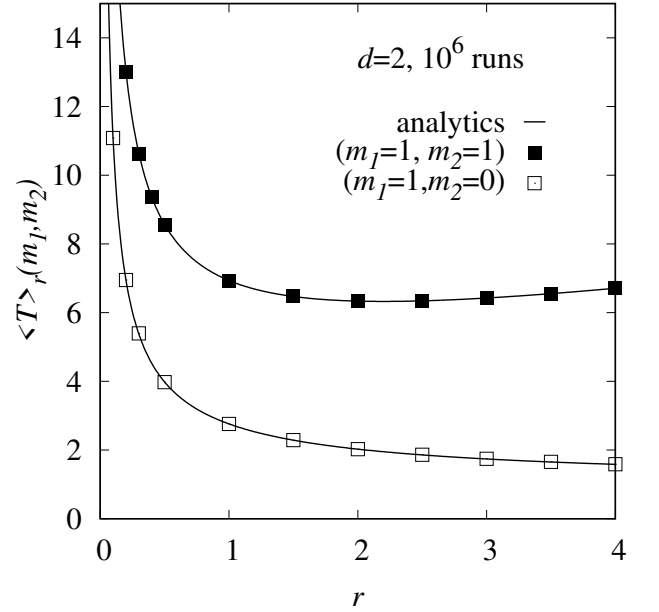


FIG. 2.  $\langle T \rangle_r(1, 0)$  and  $\langle T \rangle_r(1, 1)$  vs.  $r$  in two dimensions. The solid line shows the analytical results in Eqs. (48) and (53), while the symbols represent the numerical results. The agreement is excellent.

symmetry, this result in Eq. (48) also holds for three other starting points  $(m_1 = 0, m_2 = 1)$ ,  $(m_1 = -1, m_2 = 0)$  and  $(m_1 = 0, m_2 = -1)$ .

**The case  $(m_1 = 1, m_2 = 1)$ .** In this case, Eq. (42) reduces to

$$\langle T \rangle_r(1, 1) = \frac{1}{r} \left[ \frac{2}{\pi} \frac{K \left( \frac{16}{(4+r)^2} \right)}{\int_0^\infty dt e^{-t} \left[ I_1 \left( \frac{2t}{(r+4)} \right) \right]^2} - 1 \right]. \quad (50)$$

It turns out the integral in the denominator can again be performed using the identity, valid for  $|z| < 1$ ,

$$\int_0^\infty dt e^{-t} [I_1(zt/2)]^2 = \frac{2}{\pi z^2} [(2 - z^2)K(z^2) - 2E(z^2)], \quad (51)$$

where  $E(u)$  is the EllipticE function with argument  $u$  defined as

$$E(u) = \int_0^{\pi/2} \sqrt{1 - u \sin^2(\theta)} d\theta. \quad (52)$$

Using the identity (51) in (50) leads to the explicit result

$$\langle T \rangle_r(1, 1) = \frac{1}{r} \left[ \frac{z^2 K(z^2)}{(2 - z^2)K(z^2) - 2E(z^2)} - 1 \right], \quad (53)$$

where  $z \equiv z(r) = 4/(4+r)$ . The asymptotic behaviors are given by

$$\langle T \rangle_r(1, 1) \approx \begin{cases} -\frac{4}{r \ln(r)} & \text{as } r \rightarrow 0 \\ \frac{r}{2} + 4 + \frac{3}{r} + O(r^{-2}) & \text{as } r \rightarrow \infty. \end{cases} \quad (54)$$



In Fig. 2 we plot  $\langle T \rangle_r(1, 1)$  vs.  $r$  and compare with our direct numerical simulation, finding excellent agreement. We see that as  $r \rightarrow \infty$ , the MFPT displays a unique minimum at some  $r^*$  when this starting point  $(1, 1)$  is not a nearest neighbour of the target at  $(0, 0)$ . Again, we note that, by symmetry of the square lattice, the result in Eq. (53) also holds for three other points:  $(-1, 1)$ ,  $(-1, -1)$  and  $(1, -1)$ .

### C. Some explicit results in general dimension $d$

We now consider general dimension  $d$  and a general starting point  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$ . In this case, while the result in Eq. (33) for the MFPT  $\langle T \rangle_r(\vec{R}_0)$  is exact and easily evaluable via numerical integration, it is not easy to reduce it to an explicit formula. However, one can obtain explicitly the asymptotics behaviors of the MFPT in the two limits  $r \rightarrow \infty$  and  $r \rightarrow 0$ , as we show below.

**The limit  $r \rightarrow \infty$ .** We start with the  $r \rightarrow \infty$  which is easier. In this limit, the argument  $2t/(r+2d)$  of the Bessel functions tend to zero. We can then use the leading asymptotic behavior of  $I_{|m|}(z)$  as  $z \rightarrow 0$ , namely [27],

$$I_{|m|}(z) \approx \frac{1}{\Gamma(|m|+1)} \left(\frac{z}{2}\right)^{|m|}. \quad (55)$$

Substituting this behavior in Eq. (33) for fixed  $(m_1, m_2, \dots, m_d)$ , we get the leading order behaviors for large  $r$

$$\int_0^\infty dt e^{-t} \left[ I_0 \left( \frac{2t}{(r+2d)} \right) \right]^d \approx 1 \quad (56)$$

$$\int_0^\infty dt e^{-t} \prod_{i=1}^d I_{|m_i|} \left( \frac{2t}{(r+2d)} \right) \approx \frac{\Gamma \left( \sum_{i=1}^d |m_i| + 1 \right)}{\prod_{i=1}^d \Gamma(|m_i| + 1)} r^{-(|m_1|+|m_2|+\dots+|m_d|)}. \quad (57)$$

This leads to a rather interesting power law growth of  $\langle T \rangle_r(\vec{R}_0)$  for large  $r$  with a distance dependent exponent

$$\langle T \rangle_r(m_1, m_2, \dots, m_d) \approx \frac{\prod_{i=1}^d \Gamma(|m_i| + 1)}{\Gamma \left( \sum_{i=1}^d |m_i| + 1 \right)} r^{|m_1|+|m_2|+\dots+|m_d|-1}. \quad (58)$$

It is easy to check that in  $d = 1$  and in the special cases for  $d = 2$ , we get back previous results, respectively in Eqs.

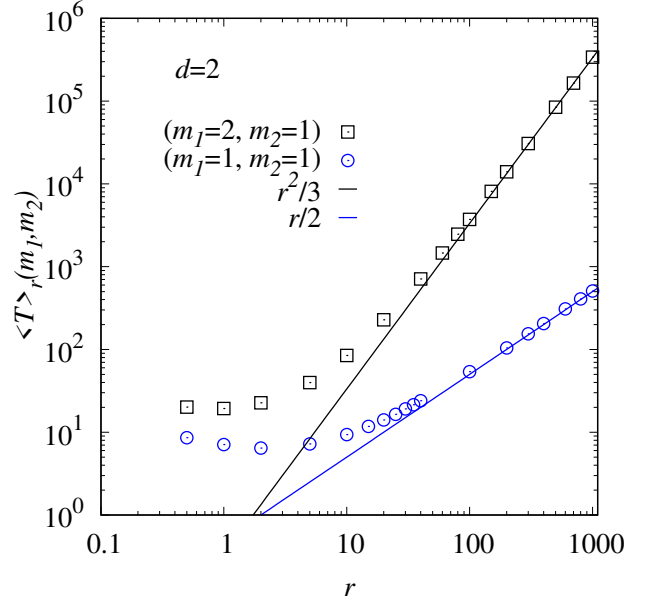


FIG. 3.  $\langle T \rangle_r(1, 1)$  vs.  $r$  and  $\langle T \rangle_r(2, 1)$  vs.  $r$  in two dimensions. The solid lines represent the asymptotic large  $r$  behavior predicted analytically in Eq. (58), while the symbols represent the numerical results. The agreement is excellent for large  $r$ .

(40), (49) and (54). Note that this result is completely inaccessible in the continuum limit.

The  $r \rightarrow \infty$  result in (58) is valid for any starting point  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$ . Interestingly, this formula predicts that for any starting point which is not a nearest neighbour of the origin, the MFPT diverges as a power law  $\langle T \rangle_r(m_1, m_2, \dots, m_d) \sim r^\phi$  as  $r \rightarrow \infty$ , with a distance dependent exponent  $\phi = |m_1| + |m_2| + \dots + |m_d| - 1$ . We have checked this result numerically in  $d = 2$  for  $(m_1 = 1, m_2 = 1)$  and  $(m_1 = 2, m_2 = 1)$ . In the former case, our result predicts a linear growth with  $r$ , namely  $\langle T \rangle_r(1, 1) \approx r/2$ , while in the latter case, it predicts a quadratic growth  $\langle T \rangle_r(2, 1) \approx r^2/3$ . Numerical results match perfectly our analytical predictions (including the prefactors  $1/2$  and  $1/3$ ), as can be seen in Fig. 3.

In contrast, if the starting point happens to be one of the  $2d$  nearest neighbours of the origin, e.g., for the case  $(m_1 = 1, 0, 0, \dots, 0)$  (or any of its  $2d$  symmetric cousins), it follows from Eq. (58) that the MFPT approaches a universal constant 1 as  $r \rightarrow \infty$ ,

$$\langle T \rangle_r(1, 0, \dots, 0) = 1 + \frac{2d-1}{r} + O\left(\frac{1}{r^2}\right) \quad \text{as } r \rightarrow \infty. \quad (59)$$

We now provide a simple physical argument for this general universal result 1 as  $r \rightarrow \infty$ . Consider the walker starting its journey from a nearest neighbour of the origin. In the limit  $r \rightarrow \infty$ , even if the walker hops from a nearest neighbour site  $\vec{R}_0$  to any other site (apart from the origin), it immediately gets reset to  $\vec{R}_0$ . Of course, if it jumps to the origin, then the process is over since the

target is found. Thus, in this high resetting limit, the dynamics of the walker gets essentially localised only to two neighboring sites, namely the starting site  $\vec{R}_0$  (nearest neighbour to the origin) and the origin itself. The effective dynamics in continuous time then reduces to the following: in a small time interval  $dt$ , the walker hops from  $\vec{R}_0$  to  $\vec{0}$  with probability  $dt$  and with the complementary probability  $(1 - dt)$ , it stays at the departure site  $\vec{R}_0$ . Thus it becomes a Poisson process with rate 1. The probability that the walker stays at the starting site  $\vec{R}_0$  up to time  $t$  is simply  $Q(\vec{R}_0, t) = e^{-t}$ . Consequently, the first-passage probability density to the origin, starting from  $\vec{R}_0$ , is simply  $F(\vec{R}_0, t) = -\partial_t Q(\vec{R}_0, t) = e^{-t}$ . Hence the MFPT is given by its first moment

$$\lim_{r \rightarrow \infty} \langle T \rangle_r(\vec{R}_0) = \int_0^\infty t e^{-t} dt = 1. \quad (60)$$

This argument is very general and leads to this universal limiting value 1 in any dimension, as long as the starting point  $\vec{R}_0$  is a nearest neighbour of the target. The fact that the global minimum of the MFPT occurs at  $r = r^* = \infty$  when the starting point is a nearest neighbour of the target is another striking result that is not captured in the continuum limit, also simply because there exists no nearest neighbour in the continuum limit.

**The limit  $r \rightarrow 0$ .** This limit, for general  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$ , turns out to be more tricky and the behaviors depend on the dimension  $d$ .

- $d = 1$ . In this case, from the explicit result in Eq.

(40), we have the leading order result as  $r \rightarrow 0$

$$\langle T \rangle_r(m_1) \approx \frac{|m_1|}{\sqrt{r}}, \quad (61)$$

valid for arbitrary  $m_1$ .

- $d = 2$ . In this case, we had exact results only in two cases: for the starting points  $(m_1 = 1, m_2 = 0)$  (and its 3 symmetric counterparts) and  $(m_1 = 1, m_2 = 1)$  (along with its three symmetric cousins). In both cases, the MFPT diverges as  $\langle T \rangle_r \approx -A/(r \ln r)$  as  $r \rightarrow 0$  where the prefactor  $A = \pi$  in the first case (see Eq. (49)), while  $A = 4$  in the second case (see Eq. (54)). We now show that for general  $(m_1, m_2)$ , the MFPT diverges as  $r \rightarrow 0$  exactly in the same way, but with a prefactor  $A(m_1, m_2)$  that depends explicitly on the starting point. More precisely, as  $r \rightarrow 0$ , we get

$$\langle T \rangle_r(m_1, m_2) \approx -\frac{A(m_1, m_2)}{r \ln r}, \quad (62)$$

where

$$A(m_1, m_2) = \int_0^\infty dt e^{-t} [I_0^2(t/2) - I_{|m_1|}(t/2) I_{|m_2|}(t/2)]. \quad (63)$$

To prove this result, we start from our exact result in Eq. (42) valid for arbitrary  $(m_1, m_2)$  that reads

$$\langle T \rangle_r(m_1, m_2) = \frac{1}{r} \left[ \frac{\int_0^\infty dt e^{-t} \left[ I_0^2\left(\frac{2t}{(r+4)}\right) - I_{|m_1|}\left(\frac{2t}{(r+4)}\right) I_{|m_2|}\left(\frac{2t}{(r+4)}\right) \right]}{\int_0^\infty dt e^{-t} I_{|m_1|}\left(\frac{2t}{(r+4)}\right) I_{|m_2|}\left(\frac{2t}{(r+4)}\right)} \right]. \quad (64)$$

Let us first consider the integral in the denominator

$$F_r(m_1, m_2) = \int_0^\infty dt e^{-t} I_{|m_1|}\left(\frac{2}{r+4}t\right) I_{|m_2|}\left(\frac{2}{r+4}t\right). \quad (65)$$

We want to extract its behavior as  $r \rightarrow 0$ . By making a change of variable  $2t/(r+4) = y$ , we get

$$F_r(m_1, m_2) = \frac{(r+4)}{2} \int_0^\infty dy e^{-ry/2} e^{-2y} I_{|m_1|}(y) I_{|m_2|}(y). \quad (66)$$

We note that since  $I_{|m|}(y) \approx e^y/\sqrt{2\pi y}$  as  $y \rightarrow \infty$  for any  $|m|$ , it is clear that the integrand in (66) behaves as  $e^{-ry/2}/(2\pi y)$  for large  $y$  and hence the integral diverges as  $r \rightarrow 0$ . To extract this leading

divergence, we divide the integral over  $y$  into two parts:  $y \in [0, \Lambda]$  and  $[\Lambda, \infty]$  where the cutoff  $\Lambda \gg 1$  but is independent of  $r$  as  $r \rightarrow 0$ . Then the leading divergence as  $r \rightarrow 0$  comes from the second interval  $y \in [\Lambda, \infty]$  where we can use the asymptotic behavior of the Bessel function  $I_{|m|}(y) \approx e^y/\sqrt{2\pi y}$ . This gives

$$F_r(m_1, m_2) \approx 2 \int_\Lambda^\infty dy \frac{e^{-ry/2}}{2\pi y} \quad (67)$$

$$= \frac{1}{\pi} \int_{\Lambda/r/2}^\infty dz \frac{e^{-z}}{z} \approx -\frac{1}{\pi} \ln r \quad (68)$$

as  $r \rightarrow 0$ . We can now use this behavior for the denominator in Eq. (64) and set  $r = 0$  in the numerator (since the integral in the numerator is

convergent as  $r \rightarrow 0$ ). This then gives the leading order  $r \rightarrow 0$  behavior in Eq. (63).

- $d > 2$ . We start from our general result (33) which is valid for any  $r$  and any starting point  $\vec{R}_0 = a\vec{m} = a(m_1, m_2, \dots, m_d)$ . In this case, by using the asymptotic behavior  $I_{|m|}(y) \approx e^y/\sqrt{2\pi y}$ , it is easy to check that both the integrals in the numerator and the denominator are convergent as  $r \rightarrow 0$ . Hence the leading  $r \rightarrow 0$  behavior of the MFPT is clearly then given by

$$\langle T \rangle_r(\vec{m}) \approx \frac{B(\vec{m})}{r}, \quad (69)$$

as  $r \rightarrow 0$  and where the prefactor is

$$\begin{aligned} B(\vec{m}) &\equiv B(\vec{R}_0) \\ &= \frac{\int_0^\infty dt e^{-t} I_0^d\left(\frac{t}{d}\right)}{\int_0^\infty dt e^{-t} \prod_{i=1}^d I_{|m_i|}\left(\frac{t}{d}\right)} - 1. \end{aligned} \quad (70)$$

Actually this prefactor  $B(\vec{R}_0)$  in Eq. (70) has a nice physical interpretation. To extract this physical meaning, let us go back to the general expression in Eq. (16). We see in Eq. (16) that in the limit  $r \rightarrow 0$  and for  $d > 2$ , one has

$$\begin{aligned} \langle T \rangle_r(\vec{R}_0) &\approx \frac{1}{r} \left[ \frac{1}{\tilde{F}_0(\vec{R}_0, 0)} - 1 \right] \\ &= \frac{1}{r} \left[ \frac{1}{\int_0^\infty F_0(\vec{R}_0, t) dt} - 1 \right]. \end{aligned} \quad (71)$$

Comparing with Eq. (69), we then identify the amplitude  $B(\vec{R}_0)$  as

$$B(\vec{R}_0) = \left[ \frac{1 - \int_0^\infty F_0(\vec{R}_0, t) dt}{\int_0^\infty F_0(\vec{R}_0, t) dt} \right]. \quad (72)$$

Now the denominator  $\int_0^\infty F_0(\vec{R}_0, t) dt < 1$  is just the hitting probability of the target in  $d > 2$ . We recall that for an ordinary random walker without resetting in  $d > 2$ , the target gets captured by the walker with probability  $\int_0^\infty F_0(\vec{R}_0, t) dt < 1$ , while with the complementary probability  $1 - \int_0^\infty F_0(\vec{R}_0, t) dt$  the walker escapes to infinity. Hence, from Eq. (72), we see that the prefactor  $B$  is just the ratio of the escape and the hitting probability for a walker in  $d > 2$  starting at  $\vec{R}_0$

$$B(\vec{R}_0) = \frac{\text{escape probability}}{\text{hitting probability}}. \quad (73)$$

Here, we thus provide an explicit expression for this ratio in Eq. (70) in terms of integrals over Bessel functions.

**Optimal resetting rate in the large dimension limit  $d \rightarrow \infty$ .** From the discussion above, it is clear that as long as the starting point is not a nearest neighbour of the target at the origin, the MFPT  $\langle T \rangle_r(\vec{R}_0)$ , as a function of  $r$  for fixed  $\vec{R}_0$ , diverges at the two limits  $r \rightarrow 0$  and  $r \rightarrow \infty$ , displaying a single minimum at the optimal value  $r^*$  which depends on both the starting position  $\vec{R}_0$  and dimension  $d$ . For a fixed starting position  $\vec{R}_0$ , it is interesting to ask how  $r^*(d)$  behaves as a function of dimension  $d$ ? To determine  $r^*$  explicitly from the exact expression (33) seems difficult. However, one can make progress in the large dimension limit. In this limit, the factor  $2/(r + 2d)$  in the argument of the Bessel functions in (33) becomes small and one can make a systematic large  $d$  expansion of (33) by using the asymptotic behavior of Bessel function for small arguments given in Eq. (55). This systematic large expansion of Eq. (33) yields, for fixed  $r$  and  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$ , the following expression for the MFPT up to leading orders in  $d$

$$\begin{aligned} \langle T \rangle_r(m_1, m_2, \dots, m_d) &\approx \\ &\beta(2d)^\alpha \left[ \frac{1}{r} + \frac{\alpha r + 1}{r} \frac{1}{2d} + \right. \\ &\left. \left( \alpha - 2 + \frac{\alpha(\alpha - 1)}{2} r \right) \frac{1}{(2d)^2} + O\left(\frac{1}{d^3}\right) \right], \end{aligned} \quad (74)$$

where

$$\alpha = \sum_{i=1}^d |m_i| \quad \text{and} \quad \beta = \frac{\prod_{i=1}^d \Gamma(|m_i| + 1)}{\Gamma(|m_1| + |m_2| + \dots + |m_d| + 1)}. \quad (75)$$

One can now minimize (74) with respect to  $r$  and one gets

$$r^*(d, (m_1, m_2, \dots, m_d)) \approx \sqrt{\frac{8}{\alpha(\alpha - 1)}} d, \quad \text{as } d \rightarrow \infty. \quad (76)$$

Let us recall that since the starting point is not a nearest neighbour of the origin, we must have  $\alpha > 1$ . In particular, if the starting point happens to be  $(2, 0, 0, \dots, 0)$ , then  $\alpha = 2$  and we have from Eq. (76)

$$r^*(d, (2, 0, 0, \dots, 0)) \approx 2d, \quad (77)$$

This result agrees well with our numerical simulations, as shown in Fig. 4. Here we have determined  $\langle T \rangle_r(2, 0, 0, \dots, 0)$  for various dimensions  $d \in [1, 50]$  and selected values of  $r$ . We have determined the optimum resetting rates  $r^*$  and minimum first-passage times  $T_0$  by fitting parabolas  $\tilde{T}(r) = a(r - r^*)^2 + T_0$  near the minimum of  $\langle T \rangle_r(2, 0, 0, \dots, 0)$ , respectively. The inset of the plot shows  $r^*$  as function of dimension  $d$ , the linear growth for large  $d$  is well visible.

In our simulations, we have also investigated the paths to the target after the last reset. With increasing dimension, at each position, the number of possible steps

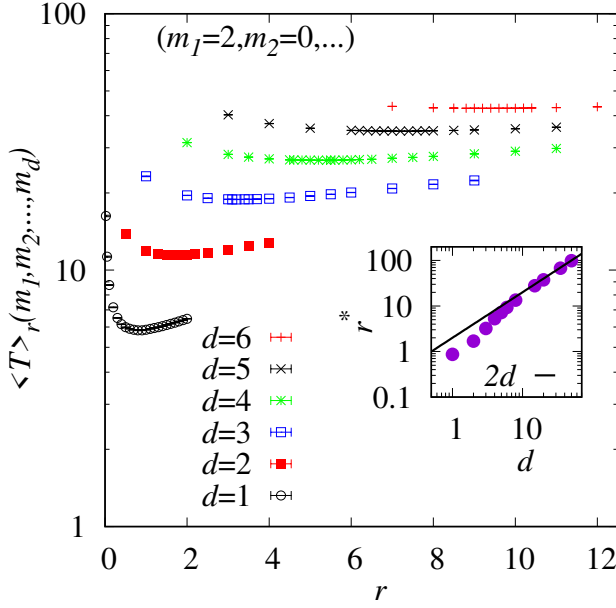


FIG. 4. The simulation results  $\langle T \rangle_r(2, 0, 0, \dots, 0)$  vs.  $r$  in dimensions  $d = 1$  to  $d = 6$ . The minima represent the optimal resetting rate  $r^*$ . The inset shows  $r^*$  as a function of  $d$  for some values  $d \leq 50$  which seems to fit at larger value of  $d$  well with the linear growth  $r^*(d) = 2d$  predicted in Eq. (77).

leading away from the target increases. Thus, the walker needs to be reset more and more to find the target as the dimension increases. This also indicates that the actual paths leading to the target after the last reset will more and more be shortest paths. We have verified this for the starting position  $\vec{m} = (2, 0, \dots, 0)$  by measuring the number of steps  $n_{\text{final}}$  the walker takes to reach the target after the last reset has happened. In Fig. 5 we show the value  $n_{\text{final}}(r^*)$  for the optimum resetting rate  $r^*$  as function of dimension  $d$ . Indeed one observes an convergence of  $n_{\text{final}}$  to the value 2. Note that also for increasing  $r$  this convergence to the value  $n_{\text{final}} = 2$  is visible, see inset of Fig. 5. This also makes sense, because with increasing resetting rate, the walker has less time to explore the space, i.e., only the shortest paths to the target will occur.

Note that we have also measured the total number  $n_{\text{hops}}$  of hops since  $t = 0$ , see Section IV, but this correlates almost perfectly with  $\langle T \rangle_r(2, 0, \dots, 0)$ , so we do not show these results here.

The observation that with increasing lattice dimension  $d$ , or increasing reset rate  $r$ , only direct paths to the target dominate, motivates a simple finite-state Markov chain model, which allows one to derive the optimum resetting rate for close targets very quickly. Since this is pedagogically interesting, we also present this approach here.

We quickly recall from the theory of Markov chains [32] how to calculate the mean-first passage time. For simplicity, we consider a discrete time Markov chain with  $N + 1$  states, where the state  $N + 1$  is the single absorbing state.

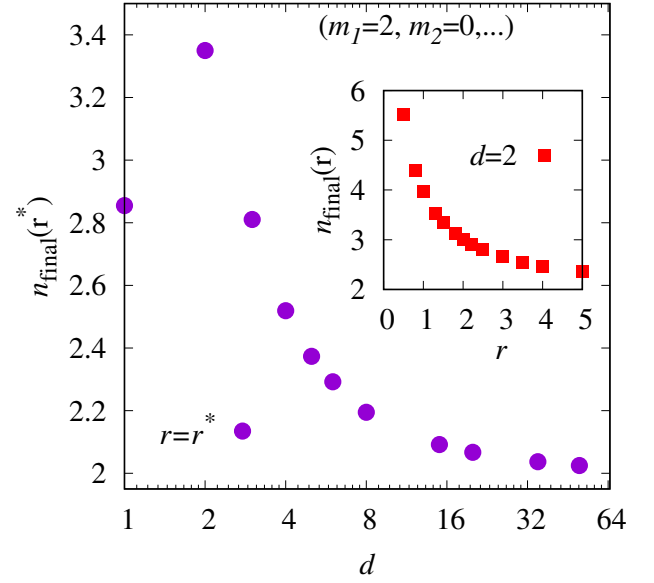


FIG. 5. The number  $n_{\text{final}}(r^*)$  of steps the walker takes since the final, i.e. successful, reset until reaching the target, measured at the optimum reset rate  $r^*$ , shown as function of the lattice dimension  $d$ . Here the target is located at  $m = (2, 0, \dots, 0)$ . The inset shows  $n_{\text{final}}$  as function of resetting rate  $r$  for dimension  $d = 2$ .

Let the column vector  $\vec{\pi}(s) = (\pi_1(s), \dots, \pi_{N+1}(s))^T$  denote the probabilities that the chain is in state  $j$  at step  $s$ . The dynamics of the Markov chain is described by transition matrix  $\mathbf{P}$  according to  $\vec{\pi}(s+1) = \mathbf{P}\vec{\pi}(s)$ , where  $p_{ij}$  is the probability that the chain moves to state  $i$  if it is in state  $j$ . Since state  $N + 1$  is absorbing, we can write  $\mathbf{P}$  as

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \vec{0} \\ \vec{u}^T & 1 \end{pmatrix}, \quad (78)$$

where the  $N \times N$  matrix  $\mathbf{Q}$  describes the transition probabilities excluding the absorbing state, i.e., among the transient states. The vector  $\vec{u}$  denotes the transition probabilities to the absorbing state. Now, the matrix  $\mathbf{P}^n$  describes the dynamics of  $n$  steps, i.e. its element  $p_{ij}^{(n)}$  the probability that the chain is after  $n$  steps in state  $i$  if it starts in state  $j$ . Correspondingly, for the matrix  $\mathbf{Q}^n$ , does the same for the set of transient states. This leads to the definition of the fundamental matrix

$$\mathbf{N} = \mathbf{I} + \mathbf{Q} + \mathbf{Q}^2 + \dots, \quad (79)$$

where  $\mathbf{I}$  is the identity matrix. The entry  $n_{ij}$  describes the expected number of times the chain has visited transient state  $i$ , given that the chains has started in state  $j$ . Therefore, the mean-first passage steps, starting at state  $j$  is given the expected number of times it has visited any transient state, i.e., by

$$\tilde{T} = \sum_{i=1}^N n_{ij}. \quad (80)$$

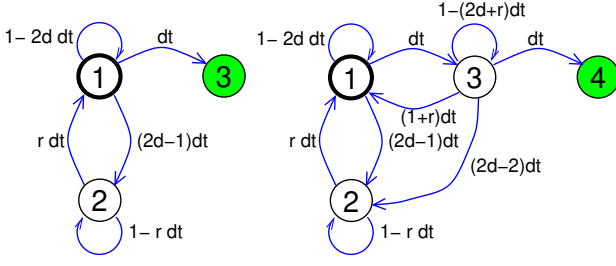


FIG. 6. Markov chains for representing lattices in  $d$  dimensions where (left) the target site (state 3) is located next to the starting and resetting site (state 1), or (right) where they are separated in one lattice direction by an intermediate site (state 3). State 2 represents all other sites. It is assumed, which holds for  $d \rightarrow \infty$  or  $r \rightarrow \infty$ , that all relevant paths to the target site do not go through state 2.

$\mathbf{N}$  can be conveniently calculated via the matrix inverse  $\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}$ , reminiscent of the well-known geometric-series identity  $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$  for  $|q| < 1$ .

We start with the case where the target is next to the starting position  $\vec{R}_0$ , which is also the resetting position. Here we approximate the dynamics by three states: state 1 is the starting and resetting site, state 3 the target site, and state 2 represents all other sites. Since we assume that only direct paths from the starting site to the target sites are relevant, we do not consider transitions between states 2 and 3. To work in probability space, we assume a small time step  $dt$ , thus, resetting occurs with probability  $rdt$ , and a move to each of the  $2d$  neighbouring sites with rate 1, i.e., probability  $dt$ . Most sites of state 2 are not neighbors of the site  $\vec{R}_0$ , so we also ignore transitions to state 1 except resetting. The states and all transitions are displayed in Fig. 6. The corresponding transition matrix reads as follows:

$$\mathbf{P} = \begin{pmatrix} 1 - 2d dt & rdt & 0 \\ (2d - 1)dt & 1 - rdt & 0 \\ dt & 0 & 1 \end{pmatrix}, \quad (81)$$

where  $\mathbf{Q}$  is the upper left  $2 \times 2$  matrix. The calculation of  $\mathbf{N}$  can be performed simply by using an symbolic linear algebra package. The calculation of the first-passage time via Eq. (80) and converting to continuous time by  $T = \tilde{T}dt$  yields

$$\langle T \rangle_r(1, 0, \dots) = 1 + \frac{2d - 1}{r} \quad (d \rightarrow \infty, r \rightarrow \infty) \quad (82)$$

Thus, the optimum resetting rate is  $r \rightarrow \infty$ , which means the walker stays on the starting site until it moves in one step to the target site. This convergence to 1 matches the  $r \rightarrow \infty$  behavior from Eq. (59), which is  $d = 2$  functional form displayed in Eq. (49).

When having the starting site  $\vec{m} = (2, 0, \dots, 0)$ , represented again by state 1, two sites away from the target site, here state 4, we have to include  $\vec{m}' = (1, 0, \dots, 0)$

as intermediate site explicitly, here state 3. The walker will move with rate 1 from the starting site to  $\vec{m}'$  with rate 1. It can move back either by a hop with rate 1 or reset with rate  $r$ , yielding  $1 + r$  as total rate. It may move to the target site with rate 1. Thus, it may move to other neighbouring sites, represented by state 2, with rate  $2d - 2$ . Again, not considering transitions between all other sites and the target or intermediate sites, and transforming the rates to probabilities by multiplying with  $dt$ , the transition matrix reads as follows:

$$\mathbf{P} = \begin{pmatrix} 1 - 2d dt & rdt & (1 + r)dt & 0 \\ (2d - 1)dt & 1 - rdt & (2d - 2)dt & 0 \\ dt & 0 & 1 - (r + 2d)dt & 0 \\ 0 & 0dt & 0 & 1 \end{pmatrix}, \quad (83)$$

Obtaining  $\mathbf{Q}$ , inverting  $\mathbf{I} - \mathbf{Q}$  and obtaining  $\tilde{T}dt$  via Eq. (80) results in

$$\langle T \rangle_r(2, 0, \dots) = r + 4d + \frac{4d^2 - 2}{r} \quad (d \rightarrow \infty, r \rightarrow \infty) \quad (84)$$

For  $d = 2$  this yields  $r + 8 + 6/r$  which is twice the result of Eq. (54) for  $\vec{m} = (1, 1)$ , which makes sense, because for that case there are two shortest path to the target, i.e., half the time is needed on average. This function exhibits a minimum, which is obtained by setting  $\frac{dT}{dr}(r^*) = 0$ , i.e.,  $r^* = \sqrt{4d^2 - 2} \rightarrow 2d$  for large values of the dimension  $d$ , compatible with the previous result of Eq. (77).

In summary, we find that in general dimension  $d$ , if the starting point  $m$  is not a nearest neighbour of the target at the origin, then the MFPT in Eq. (33), as a function of  $r$ , diverges at both ends  $r \rightarrow 0$  and  $r \rightarrow \infty$ , and displays a minimum at an optimal value  $r^*(\vec{R}_0)$  that depends explicitly on the starting point  $\vec{R}_0$  and the dimension  $d$ . In Fig. 4 we show the MFPT as a function of  $r$ , for the starting point  $\vec{R}_0 = (m_1 = 2, m_2 = 0, m_3 = 0, \dots, m_d = 0)$ , for dimensions  $d = 1, 2, 3, 4, 5$ . We see that  $r^*$  increases with increasing  $d$ . In principle, one can numerically determine this optimal value by setting to zero the derivative of Eq. (33) with respect to  $r$  and  $r = r^*$  and then determining this root  $r^*$  numerically. For example, for  $d = 1$  and with  $m_1 = 2$ , this gives  $r^* \approx 0.828489$  which matches very well with the simulation value. We have shown that in the large  $d$  limit, one can prove analytically that  $r^*(d) \approx 2d$  when the starting position is  $\vec{R}_0 = (m_1 = 2, m_2 = 0, m_3 = 0, \dots, m_d = 0)$ . In contrast, if the starting point happens to be a nearest neighbour of the target, then the MFPT decreases monotonically with increasing  $r$  and achieves its lowest value 1 as  $r \rightarrow \infty$ , see Eq. (59). Hence, in this case, the optimal value  $r^*$  is infinite, a result somewhat unexpected.

## VI. NONEQUILIBRIUM STATIONARY STATE IN THE ABSENCE OF A TARGET

So far, we have focused on the MFPT to a target placed at the origin for a resetting random walker on a  $d$ -dimensional lattice. In the absence of a target, as in the continuous space case, the resetting to the initial position  $\vec{R}_0$  with a constant rate  $r$  drives the system into a nonequilibrium stationary state (NESS) at long times. In this section, we present the position distribution in the NESS for such a walker on the lattice. Since there is no target, the system is translationally invariant on the lattice, and without any loss of generality, we can set the starting (and the resetting) position  $\vec{R}_0 = \vec{0}$ . Then we just need to compute  $P_r(\vec{R}, \vec{0}, t)$  in the long time limit. This probability evolves via the Fokker-Planck equation (3) discussed in Section II. One can easily solve this linear equation. However, the solution can also be obtained by employing a renewal equation approach. One can express the probability in the presence of resetting in terms of the probability without resetting via

$$P_r(\vec{R}, \vec{0}, t) = e^{-rt} P_0(\vec{R}, \vec{0}, t) + r \int_0^t d\tau e^{-r\tau} P_0(\vec{R}, \vec{0}, \tau). \quad (85)$$

The first term describes the ‘no resetting’ event in the interval  $[0, t]$  with the walker freely propagating from  $\vec{0}$  to  $\vec{R}$  in time  $t$  without resetting. The second term describes events with one or more resettings. In this case, let  $t - \tau_l$  denotes the epoch at which the last resetting event took place before  $t$ . Then  $\tau_l$  is simply the time interval between  $t$  and the last resetting event before  $t$ . Since after the last resetting, the walker propagates freely from time  $t - \tau_l$  to  $t$ , we have the factor  $P_0(\vec{R}, \vec{0}, \tau_l)$  inside the integral on the rhs of (85). The probability that there is no resetting event in the interval  $[t - \tau_l, t]$  and  $t$ , preceded by a resetting event just at  $t - \tau_l$  is simply  $r e^{-r\tau_l} d\tau_l$ . Multiplying the two and integrating over all possible values of  $\tau_l \in [0, t]$  gives the second term in (85). One can check that the solution (85) indeed satisfies the Fokker-Planck equation (3) at all times with the correct initial and boundary conditions. Finally, in the limit  $t \rightarrow \infty$  the first term drops out and we obtain the stationary position distribution in the NESS

$$P_r^{\text{NESS}}(\vec{R}) = P_r(\vec{R}, \vec{0}, \infty) = r \int_0^\infty d\tau e^{-r\tau} P_0(\vec{R}, \vec{0}, \tau) = r \tilde{P}_0(\vec{R}, \vec{0}, r), \quad (86)$$

where  $\tilde{P}_0(\vec{R}, \vec{0}, r)$  is just the Laplace transform of the position distribution without resetting and has already been computed explicitly in (32). Using this result in (86) gives us the exact position distribution in the NESS

$$P_r^{\text{NESS}}(\vec{R} = a(m_1, m_2, \dots, m_d)) = \frac{r}{r + 2d} \int_0^\infty dt e^{-t} \prod_{i=1}^d I_{|m_i|} \left( \frac{2t}{(r + 2d)} \right). \quad (87)$$

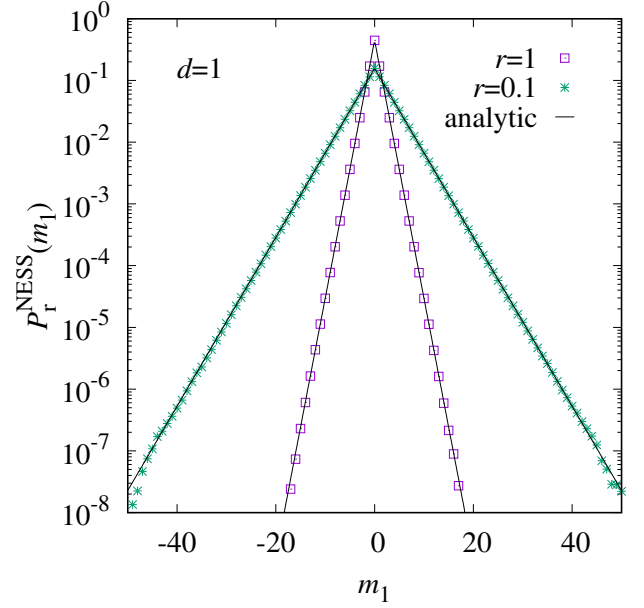


FIG. 7. Non-equilibrium steady-state density for dimension  $d = 1$  from Eq. (88), shown as lines, compared to the numerically obtained density, shown as symbols, for two cases  $r = 1$  and  $r = 0.1$ .

For example, in  $d = 1$ , using the identity (38), we get

$$P_r^{\text{NESS}}(R = a m_1) = \frac{r}{\sqrt{r^2 + 4r}} \left[ \frac{r + 2 - \sqrt{r^2 + 4r}}{2} \right]^{|m_1|}. \quad (88)$$

One can check, by summing over  $m_1$ , that  $P_r^{\text{NESS}}(R)$  is normalized to unity. Eq. (88) is compared in Fig. 7 to the numerical density for resetting rates  $r = 0.1$  and  $1$ , respectively. The numerical results were obtained from  $10^6$  independent runs up to time  $t = 10^3$ , respectively.

The expression in Eq. (87) is the main result of this section. Following exactly the same method as in Section III and the Appendix A, one can show that in the limit of lattice spacing  $a \rightarrow 0$ , one recovers the continuum results found in Ref. [7] in general dimension  $d$ . We do not repeat the calculation here. For example, in  $d = 1$ , one obtains by taking  $a \rightarrow 0$  limit in Eq. (88)

$$P_r^{\text{NESS}}(x = a m_1) \rightarrow \frac{a}{2} \tilde{r} e^{-\sqrt{\tilde{r}}|x|}, \quad (89)$$

where  $\tilde{r} = a^2 r$  is the rescaled resetting rate. Using the fact that the lattice position distribution is just  $a$  times the position density in the continuum, one then recovers the well known result in the continuum in  $d = 1$  (upon setting the diffusion constant  $D = 1$ ) [1]. Note however that, as in the case of the MFPT in earlier sections, the continuous space result requires taking the scaling limit  $r \rightarrow 0$ ,  $R \rightarrow \infty$ , while keeping the product  $\sqrt{r} R$  fixed. This is true in all dimensions. However, the lattice result in (87) is valid for arbitrary  $r$  and arbitrary  $\vec{R}$  and hence has richer informations that are not accessible within the continuum approach.

As an example, we consider again the limit  $r \rightarrow \infty$ , i.e., large resetting rate. Taking this limit in Eq. (87) by following the same procedure as in Section VC, one gets

$$P_r^{\text{NESS}}(\vec{R} = a(m_1, m_2, \dots, m_d)) \approx \frac{\Gamma\left(\sum_{i=1}^d |m_i| + 1\right)}{\prod_{i=1}^d \Gamma(|m_i| + 1)} r^{-(|m_1| + |m_2| + \dots + |m_d|)}. \quad (90)$$

This means that as one goes further away from the origin, the position distribution decays exponentially as  $\sim \exp[-\ln r (|m_1| + |m_2| + \dots + |m_d|)]$  and the decay length scales for large  $r$  as  $\sim 1/(\ln r)$ . This large  $r$  result can not be captured in the continuum limit.

## VII. CONCLUSION

In this paper we have studied a single particle diffusing on a  $d$ -dimensional lattice starting from a fixed initial position  $\vec{R}_0$  and stochastically resetting to  $\vec{R}_0$  with rate  $r$ . Our main focus was on the mean first-passage time (MFPT) to a target at the origin, as a function of the resetting rate  $r$  and the initial position  $\vec{R}_0$ . We first derived a general formula relating the MFPT to the lattice Green's function and then using it, found an exact formula for the MFPT on a  $d$ -dimensional hypercubic lattice. The previous known results were for diffusion in continuous space which we recover from our more general formula in the scaling limit:  $r \rightarrow 0$ ,  $R_0 \rightarrow \infty$ , but keeping  $\sqrt{r} R_0$  fixed. However, our exact formula lets us explore a much wider parameter space, i.e., for different values of  $r$  as well as  $\vec{R}_0$ , going beyond the continuum theory. One such interesting result is that the MFPT, as a function of  $r$  for fixed  $\vec{R}_0$ , diverges as  $r \rightarrow 0$  and  $r \rightarrow \infty$  with a minimum in between, provided the starting point is not a nearest neighbour of the target. In this case, the MFPT diverges as a power law as  $r \rightarrow \infty$ , i.e.,  $\langle T \rangle_r(\vec{R}_0) \sim r^\phi$ , but with an exponent  $\phi = (|m_1| + |m_2| + \dots + |m_d|) - 1$  that varies with the starting position  $\vec{R}_0 = a(m_1, m_2, \dots, m_d)$  (here  $a$  is the lattice spacing). In contrast, if the walker starts from a nearest neighbour of the target, then the MFPT decreases monotonically with increasing  $r$ , approaching a universal limiting value 1 as  $r \rightarrow \infty$ , indicating that the optimal resetting rate in this case is infinity. These interesting results on a lattice are not captured by the continuum theory. We have also performed high precision numerical simulations on hypercubic lattices up to 50 dimensions and find excellent matching with our analytical predictions. For targets close to the starting points, in the limit of high dimensions or large resetting rates,  $\langle T \rangle_r(\vec{R}_0)$  can also be obtained by a simple pedagogical Markov chain formulation.

Finally, in the absence of a target, we have also computed exactly the position distribution in the nonequi-

librium stationary state that also displays interesting regimes for large  $r$  which are not captured by the continuum limit.

Our results thus demonstrate, within the context of a single particle diffusion, that there are very interesting lattice effects in the presence of a nonzero resetting rate  $r$ . Since the continuum theory requires scaling the resetting rate  $r \rightarrow ra^2$  to zero as the lattice spacing  $a \rightarrow 0$ , it misses many effects, in particular for large resetting rate  $r$ , that are captured by the lattice computations. Since many models of stochastic resetting have been studied in continuous space in the literature, it would be natural to study them on lattices to see if there are interesting effects due to the lattice structure of the underlying space.

There are other directions in which this work can be extended. Here we focused on a single free particle on a  $d$ -dimensional lattice. Recent studies have shown that if  $N$  independent particles diffuse in continuous space starting from a common initial position  $\vec{R}_0$  and get simultaneously reset to  $\vec{R}_0$ , the system is driven at long times into a nonequilibrium stationary state with dynamically emergent correlations between particles [29–31]. Using techniques developed in this paper, it would be interesting to compute the stationary state for this simultaneously resetting independent  $N$ -particle system that diffuse on a lattice. One may expect new interesting lattice effects on the mutual all-to-all correlations between the particles in the stationary state, in particular in the limit  $r \rightarrow \infty$  which is not captured by the continuum theory.

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### Appendix A: The continuum limit of the lattice MFPT in Eq. (33)

To take the continuum limit of Eq. (33), it is convenient to start from the most general expression for MFPT in (19) that reads

$$\langle T \rangle_r(\vec{R}_0) = \frac{1}{r} \left[ \frac{\tilde{P}_0(\vec{0}, \vec{0}, r)}{\tilde{P}_0(\vec{0}, \vec{R}_0, r)} - 1 \right]. \quad (A1)$$



For  $\tilde{P}_0(\vec{0}, \vec{R}_0, r)$ , it is now convenient to use the integral representation (30) which reads

$$\tilde{P}_0(\vec{0}, \vec{R}_0, r) = \frac{1}{(r+2d)} \int_0^\infty dt e^{-t} \prod_{i=1}^d \int_{-\pi}^\pi \frac{dk_i}{2\pi} \exp \left[ \frac{2t}{(r+2d)} \cos(k_i) \right] e^{-i \vec{k} \cdot \vec{R}_0/a}. \quad (\text{A2})$$

Next we make the change of variable  $k_i = a q_i$ . This gives

$$\tilde{P}_0(\vec{0}, \vec{R}_0, r) = \frac{a^d}{(r+2d)} \int_0^\infty dt e^{-t} \prod_{i=1}^d \int_{-\pi/a}^{\pi/a} \frac{dq_i}{2\pi} \exp \left[ \frac{2t}{(r+2d)} \cos(a q_i) \right] e^{-i \vec{q} \cdot \vec{R}_0}. \quad (\text{A3})$$

As  $a \rightarrow 0$ , we expand  $\cos(a q_i) \approx 1 - a^2 q_i^2/2$  to rewrite this as

$$\tilde{P}_0(\vec{0}, \vec{R}_0, r) \approx \frac{a^d}{(r+2d)} \int_0^\infty dt e^{-rt/(r+2d)} \prod_{i=1}^d \int_{-\pi/a}^{\pi/a} \frac{dq_i}{2\pi} e^{-\frac{a^2 t}{(r+2d)} q_i^2} e^{-i \vec{q} \cdot \vec{R}_0}. \quad (\text{A4})$$

Since  $a \rightarrow 0$ , the limits of the integrals can be pushed to  $\mp\infty$  to leading order and one can make use of the identity

$$\prod_{i=1}^d \int_{-\infty}^\infty \frac{dq_i}{2\pi} e^{-\sigma^2 q_i^2/2 - i \vec{q} \cdot \vec{R}_0} = \frac{1}{(\sigma \sqrt{2\pi})^d} e^{-R_0^2/2\sigma^2}. \quad (\text{A5})$$

This gives

$$\tilde{P}_0(\vec{0}, \vec{R}_0, r) \approx \frac{(r+2d)^{d/2-1}}{(4\pi)^{d/2}} \int_0^\infty \frac{dt}{t^{d/2}} \exp \left[ -\frac{r}{r+2d} t - \frac{(r+2d) R_0^2}{4 a^2 t} \right]. \quad (\text{A6})$$

We next perform the integral using the following identity [27]

$$\int_0^\infty x^{\nu-1} e^{-\beta/x - \gamma x} dx = 2 \left( \frac{\beta}{\gamma} \right)^{\nu/2} K_\nu \left( \sqrt{4\beta\gamma} \right). \quad (\text{A7})$$

Using this identity in (A6) and setting  $r = a^2 \tilde{r}$  gives finally

$$\tilde{P}_0(\vec{0}, \vec{R}_0, r) \approx \frac{R_0^\nu}{(2\pi)^{d/2} a^{2\nu} \tilde{r}^{\nu/2}} K_\nu \left( R_0 \sqrt{\tilde{r}} \right) \quad (\text{A8})$$

where  $\nu = 1 - \frac{d}{2}$ . This gives us the denominator in Eq. (A1). To obtain the numerator we just take the limit  $R_0 \rightarrow 0$  in (A8) by using the small argument asymptotics of the Bessel function in Eq. (36). Putting these results together in Eq. (A1) and rescaling  $T = \tilde{T}/a^2$ , we obtain our desired result in Eq. (34).

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- [1] M.R. Evans, S.N. Majumdar, *Diffusion with Stochastic Resetting*, Phys. Rev. Lett. **106**, 160601 (2011).
  - [2] M.R. Evans, S.N. Majumdar, *Diffusion with Optimal Resetting*, J. Phys. A-Math. & Theor. **44**, 435001 (2011).
  - [3] M. R. Evans, S. N. Majumdar, G. Schehr, *Stochastic resetting and applications*, J. Phys. A. : Math. Theor. **53**, 193001 (2020).
  - [4] A. Pal, S. Kostinski, S. Reuveni, *The inspection paradox in stochastic resetting*, J. Phys. A: Math. Theor. **55**, 021001 (2022).
  - [5] S. Gupta, A. M. Jayannavar, *Stochastic resetting: A (very) brief review*, Frontiers in Physics **10**, 789097 (2022).
  - [6] S. Gupta, A. Nagar, *Stochastic resetting in interacting*

- 
- particle systems: A review*, J. Phys. A: Math. Theor. **56**, 283001 (2023).
  - [7] M. R. Evans, S. N. Majumdar, *Diffusion with resetting in arbitrary spatial dimension*, J. Phys. A: Math. Theor. **47**, 285001 (2014).
  - [8] O. Tal-Friedman, A. Pal, A. Sekhon, S. Reuveni, Y. Roichman, *Experimental realization of diffusion with stochastic resetting*, J. Phys. Chem. Lett. **11**, 7350 (2020).
  - [9] B. Besga, A. Bovon, A. Petrosyan, S. N. Majumdar, and S. Ciliberto, *Optimal mean first-passage time for a Brownian searcher subjected to resetting: experimental and theoretical results*, Phys. Rev. Res. **2**, 032029(R) (2020).
  - [10] F. Faisant, B. Besga, A. Petrosyan, S. Ciliberto, S. N.



- Majumdar, *Optimal mean first-passage time of a Brownian searcher with resetting in one and two dimensions: experiments, theory and numerical test*, J. Stat. Mech. 113203 (2021).
- [11] D. Boyer and C. Solis-Salas, *Random walks with preferential relocations to places visited in the past and their application to biology*, Phys. Rev. Lett. **112**, 240601 (2014).
  - [12] S. N. Majumdar, S. Sabhapandit, G. Schehr, *Random walk with random resetting to the maximum position*, Phys. Rev. E **92**, 052126 (2015).
  - [13] A. Falcon-Cortes, D. Boyer, L. Giuggioli, S. N. Majumdar *Localization transition induced by learning in random searches*, Phys. Rev. Lett. **119**, 140603 (2017).
  - [14] D. Boyer, A. Falcon-Cortes, L. Giuggioli, S. N. Majumdar, *Anderson-like localization transition of random walks with resetting*, J. Stat. Mech., 053204 (2019).
  - [15] O. L. Bonemo, A. Pal, *First passage under restart for discrete space and time: Application to one-dimensional confined lattice random walks*, Phys. Rev. E **103**, 052129 (2021).
  - [16] L. N. Christophorov, *Resetting random walks in one-dimensional lattices with sinks*, J. Phys. A: Math. Theor. **55**, 155006 (2022).
  - [17] A. Barbini, L. Giuggioli, *Lattice random walk dynamics with stochastic resetting in heterogeneous space*, J. Phys. A: Math. Theor. **57**, 425001 (2024).
  - [18] A. P. Riascos, D. Boyer, P. Herringer, J. L. Mateos, *Random walks on networks with stochastic resetting*, Phys. Rev. E **101**, 062147 (2020).
  - [19] M. Biroli, F. Mori, and S. N. Majumdar, *Number of distinct sites visited by a resetting random walker*, J. Phys. A: Math. Theor. **55**, 244001 (2022).
  - [20] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, 2001).
  - [21] A. J. Bray, S. N. Majumdar, G. Schehr, *Persistence and first-passage properties in nonequilibrium systems*, Adv. in Phys. **62**, 225 (2013).
  - [22] L. Kusmierz, S. N. Majumdar, S. Sabhapandit, G. Schehr, *First order transition for the optimal search time of Lévy flights with resetting*, Phys. Rev. Lett., **113**, 220602 (2014).
  - [23] S. Reuveni, *Optimal stochastic restart renders fluctuations in first passage times universal*, Phys. Rev. Lett. **116**, 170601 (2016).
  - [24] A. J. Guttmann, *Lattice Green's functions in all dimensions*, J. Phys. A: Math. Theor. **43**, 305205 (2010).
  - [25] G. H. Vineyard, *The number of distinct sites visited in a random walk on a lattice*, J. Math. Phys. **4**, 1191 (1963).
  - [26] E. W. Montrol and G. H. Weiss, *Random Walks on Lattices. II*, J. Math. Phys. **6**, 167 (1965).
  - [27] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 7th edition, Academic Press (San Diego, 2007).
  - [28] A. K. Hartmann, *Big Practical Guide to Computer Simulations* (World Scientific, Singapore 2015).
  - [29] M. Biroli, H. Larralde, S. N. Majumdar, G. Schehr, *Extreme Statistics and Spacing Distribution in a Brownian Gas Correlated by Resetting*, Phys. Rev. Lett., **130**, 207101 (2023).
  - [30] M. Biroli, S. N. Majumdar, G. Schehr, *Critical number of walkers for diffusive search processes with resetting*, Phys. Rev. E, **107**, 064141 (2023).
  - [31] M. Biroli, H. Larralde, S. N. Majumdar, G. Schehr, *Exact extreme, order and sum statistics in a class of strongly correlated system*, Phys. Rev. E **109**, 014101 (2024).
  - [32] C. M. Grinstead and J. L. Snell, *Introduction to Probability*, (American Mathematical Society, 2012)