Conformal Invariance in Two-Dimensional Ising Spin Glasses

C. Amoruso,[1] A. K. Hartmann,[2] M. B. Hastings,[3] and M. A. Moore[1]

[1] *School of Physics and Astronomy, University of Manchester, Manchester M13 9PL, U. K.*
[2] *Institut für Theoretische Physik, Universität Göttingen, Friedrich-Hund-Platz 1, 37077, Göttingen, Germany*
[3] *Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545*

(Dated: July 17, 2006)

We present evidence and arguments that the techniques of conformal field theory might be applicable to two-dimensional Ising spin glasses with Gaussian bond distributions. Our main evidence is numerical and it is that certain domain wall distributions in one geometry can be related to that in a second geometry by a conformal transformation. We also discuss briefly the assumption that the domain walls are actually stochastic Loewner (SLE) processes. If so, the fractal dimension d_f will be related to their interface energy exponent θ and we suggest that the relation is $d_f - 1 = 3/[4(3+\theta)]$, which is consistent with the commonly quoted values $d_f \approx 1.27$ and $\theta \approx -0.28$.

PACS numbers: 11.25.Hf, 75.50.Lk, 05.50.+q

The powerful tools provided by conformal field theory (CFT) have permitted the determination of the exponents associated with most two-dimensional (2d) critical phenomena [1]. Unfortunately, CFT has not to date provided any results on strongly disordered systems like spin glasses. Recently, numerical calculations of the conformal charge for disordered random-bond Ising systems at the ferromagnet $\leftrightarrow$ paramagnet transition have been obtained [2, 3]. Nevertheless, these calculations are based on the assumption of conformal invariance (CI) of the system, but they do not prove CI itself. Here, we consider 2d spin glasses, which are widely believed to have only a phase transition at zero temperature i.e. $T_c = 0$. However, as the temperature of the system T is reduced to zero, the correlation length ξ increases to infinity [4, 5]. If the Hamiltonian of the system is $H = -\sum_{<ij>} J_{ij} S_i S_j$, with $S_i = \pm 1$, $\xi(T) \sim T^{-\nu}$ if the nearest-neighbor bond distribution J_{ij} is continuous, as in the Gaussian distribution. The exponent $\nu = -1/\theta$, where θ is the exponent which describes how the energy ΔE of a domain wall (DW) which crosses a system of linear extent depends upon L: $\Delta E \sim L^{\theta}$. According to numerical studies [6, 7] θ lies between -0.28 and -0.29. The DW has a fractal dimension $d_f \approx 1.27 \pm 0.01$ [8]. The $\pm J$ bond distribution behaves differently. θ in this case is zero [9]. While the disorder present in spin glasses means that at a local level there is not even translational invariance, the existence of a diverging lengthscale such as $\xi(T)$ suggests that on long lengthscales such microscopic features could become irrelevant and that possibly even CI might arise. It is the chief purpose of this Letter to provide numerical evidence that this is indeed the case.

The main part of this letter is a numerical study of whether there is conformal invariance of the DW distribution in 2d spin glasses. Within our numerical accuracy, CI does seem to hold in the thermodynamic limit. To our knowledge, this is the first strongly disordered system, where evidence for CI has been provided. In the reminder of this letter, we discuss whether DWs in 2d can be described by a so-called stochastic Loewner evolution (SLE) processes [10], i.e. a special stochastic differential equation, and we suggest a relationship between θ and d_f, hence these exponents seem not to be independent.

CI of the DW distribution implies that given two geometries related by a conformal transformation, then the probability of finding the DW in a given configuration in one geometry is related to the probability of finding the conformally transformed DW configuration in the transformed geometry. We will find a transformation $F(z)$ mapping geometry (a) onto (d) in Fig. 1. The two rectangles have periodic boundary conditions so that the left and right edges are identified, while the top and bottom edges are open i.e. they have the topology of an annulus. The two slits in (d) also have open boundary conditions, so that no bonds cross the slits. The rectangle (a) has an arbitrary aspect ratio; we can tune the distance between the slits in (d) as desired, with the aspect ratio of the right rectangle being a function of the distance between the slits and the aspect ratio of the left rectangle as given implicitly by Eq. (3). The dashed and dotted lines in each geometry are also conformally mapped to each other.

Before presenting the desired conformal mapping, we discuss the implications of CI in these geometries. First, we present the implications of conformal invariance for the probability distribution of the domain wall in these two geometries. We can measure the probability $p_1(n)$ of the domain wall crossing the dashed line n times in the rectangle geometry, (a), as well as the probability $p_1'(n)$ of the domain wall crossing the line n times in the slit geometry, (d). In Fig. 1, we also show the mapping of a DW which crosses the dashed line three times. In addition, we can measure the probability $p_2(n)$ of the domain wall crossing the dotted line n times in the rectangle geome-

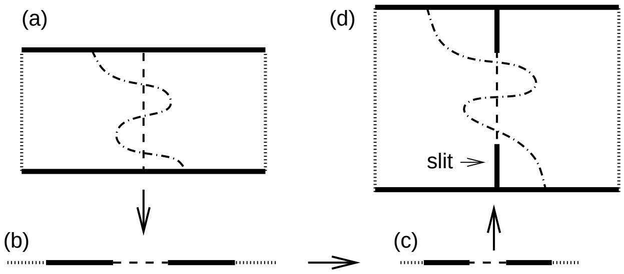

FIG. 1: Mapping between different geometries, and of the DW, shown as the dashed-dotted line.

try, as well as the probability $p_2'(n)$ of n crossings in the slit geometry. By statistical translational invariance, this probability is equal to $p_1(n)$. Thus, a naive application of CI would suggest that $p_2'(n) = p_1'(n)$. However, in the continuum limit it is not possible to distinguish between n and $n+2$ crossings of the dashed line for $n \neq 0$; if the distance between successive crossings of the dashed line in Fig. 1 is of the order of the lattice spacing, this looks at large scale like a single crossing of the dashed line. Thus, the predictions of CI are:

$$\sum_{n\,\mathrm{even}} p_1'(n) \equiv p_1' = \sum_{n\,\mathrm{even}} p_2'(n) \equiv p_2', \qquad (1)$$
$$\sum_{n\,\mathrm{odd}} p_1'(n) = \sum_{n\,\mathrm{odd}} p_2'(n),$$
$$p_1'(0) = p_2'(0).$$

These identities in Eq. (1) are the consequence of a conformal automorphism of the slit geometry which interchanges the dotted and dashed lines: $z \to F(F^{-1}(z)+\pi)$. We explicitly checked for a small rectangular system ($N = 30*30$) that $p1 = p2 \sim 0.603$ and they are equal to p_2' (quoted in tables 1 and 2) within the statistical error.

Our main test for CI is to find whether $p_1' = p_2'$.

The mapping to the slit geometry has several advantages for testing CI. Taking the dashed line small provides a strong test for CI; the naive expectation is that the DW is less likely to cross the dashed line, due to the constriction reducing the number of possible configurations, while CI instead requires these probabilities to be equal. Unlike mappings from cylinder to annulus geometries, the slits have rectangular boundaries and so are easier to implement with a finite square lattice grid. Further, the rectangles are periodic in only one direction, and have open boundary conditions in the other; the computer algorithm used here to find the ground state of the spin glass only works with this kind of boundary condition.

Constructing the Conformal Transformation— We define the rectangle (a) to have width 2π and height h, and thus aspect ratio $h/2\pi$. The horizontal direction as plotted is the real coordinate running from $-\pi$ to π while the vertical is the imaginary coordinate, which runs from $-ih/2$ to $ih/2$. We define $\tau(h) = 2\pi i/h$, and $\lambda(\tau)$ to be the modular lambda elliptic function (all conventions for elliptic functions follow those used in Mathematica). We define a function $g(z)$ to map from (a) to (b) in Fig. 1 by

$$g_h(z) = \mathrm{sn}(2izK(\lambda(\tau(h)))/h|\lambda(\tau(h))), \qquad (2)$$

where K is the complete elliptic integral of the first kind and sn is the Jacobi elliptic function [11]. This function g maps the dashed and dotted lines as shown. The dotted lines extend off to infinity and all lines lie on the real axis. The upper and lower lines of the rectangle are mapped to the two solid lines, which have endpoints at ± 1 and $\pm 1/\sqrt{\lambda(\tau(h))}$.

The mapping from (b) to (c) is simply $z \to sz$, for some parameter $0 < s \leq 1$; smaller values of s produces deeper cuts into the rectangle in (d). In (c), the endpoints of the lines are at $\pm s$ and $\pm s/\sqrt{\lambda(\tau)}$. We then determine the height h' of the rectangle in (d) such that

$$1/\sqrt{\lambda(\tau(h'))} = s/\sqrt{\lambda(\tau(h))}. \qquad (3)$$

The final mapping from (c) to (d) is $z \to g_{h'}^{-1}(z)$. This maps the portion of the solid lines in (c) between ± 1 and $\pm 1/\sqrt{\lambda(\tau(h'))}$ onto the upper and lower borders of the rectangle in (d), while the portion of the solid lines in the third geometry between $\pm s$ and ± 1 are mapped onto the slits. Thus, the full mapping from (a) to (d) is

$$F(z) = g_{h'}^{-1}(s g_h(z)), \qquad (4)$$

and the endpoints of the slits in (d) are located at $F(\pm ih/2)$.

The effect of this transformation is to increase the aspect ratio of the rectangle from its initial value s if $s < 1$ so $h' > h$. Some typical values are: for $s = 0.9$, the initial height h is $2\pi s \approx 5.6549$, while the final height $h' \approx 7.00545$ and the final aspect ratio is $h'/(2\pi) \approx 1.11495$. The distance between the slit endpoints divided by the final height h' is approximately 0.646815. For $s = 0.95$ this latter ratio is approximately 0.754943 with $h' \approx 6.60216$, while for $s = 0.85$ the same ratio is 0.557043 and $h' \approx 7.54534$.

We proceed by first finding the ground state of the system, using a mapping to a graph-theoretical problem, the minimum-weight perfect matching problem [12]. Domain walls were created by flipping the signs of the horizontal bonds in a column. Because of the periodicity in the horizontal direction, this induces a DW across the system and it is the crossings of the dashed central line and the dotted 'end' line which we study. A DW is best defined as a walk on the lattice dual to the original lattice, and the dotted and dashed lines are also lines of this dual lattice.

In Tables 1 and 2 are displayed the values of p_1' and p_2' and the probabilities of zero crossings, $p_1'(0)$ and $p_2'(0)$ for $s = 0.95$ and $s = 0.90$ for various 'sizes'. Thus $30*32$

means that the system studied is rectangular with 30 spins on each horizontal line (the direction in which the system is periodic) and 32 spins on each vertical line. The next number 8 is the number of rows cut by slits (4 at the top of the system, 4 at the bottom of the system) indicating that there are $(32-8)$ rows not cut by slits. Notice that for small sizes it is not possible to find integers to get the sizes precisely correct e.g. the aspect ratio $32/30 = 1.06666$ which is not the value $h'/2\pi \approx 1.05077$ which we are aiming to study. However, as the sizes increase the problem tends to diminish.

TABLE 1: Approach to CI for $s = 0.95$

SIZE	p'_1	p'_2	$p'_1(0)$	$p'_2(0)$	samples
$30*32$; 8	0.624	0.603	0.463	0.409	20000
$52*55$; 14	0.618	0.609	0.424	0.392	10000
$76*80$; 20	0.616	0.615	0.417	0.385	10000
$98*103$; 26	0.617	0.615	0.414	0.385	7000

TABLE 2: Approach to CI for $s = 0.90$

SIZE	p'_1	p'_2	$p'_1(0)$	$p'_2(0)$	samples
$30*33$; 12	0.639	0.604	0.494	0.407	20000
$50*56$; 20	0.623	0.612	0.448	0.399	10000
$98*109$; 38	0.620	0.611	0.421	0.384	6000
$124*138$; 48	0.612	0.610	0.411	0.381	6000

For each probability p in these tables $\sqrt(p(1-p)/N_s)$ is its standard deviation, where N_s is the number of samples i.e. bond realizations averaged over. As the size increases so the continuum limit is approached, the closer p'_1 and p'_2 become, implying that the distribution of the DWs is conformal. In Fig. 2 we have plotted p'_1 versus $1/L_1^{d_f-1}$ and p'_2 versus $1/L_2^{d_f-1}$. (We have no proof that this is the way that p'_1 and p'_2 approach their asymptotic limit, but this dependence will be partly motivated below). Again one can see that in the continuum limit CI seems to hold. L_1 is the number of rows not cut by the slit and L_2 is the total number of rows in the system. Thus for the 30*32 system size, $L_1 = 24$ and $L_2 = 32$.

Unfortunately the probabilities for zero crossings $p'_1(0)$ and $p'_2(0)$ appear to approach each other very slowly (if at all). In an attempt to understand this behavior, we have studied the probabilities of n crossings, $p'_1(n)$, of the central (dashed) line, and $p'_2(n)$ of the right (dotted) line. $p'_1(n)$ has a scaling dependence on the number of crossings n as $1/L_1^{d_f-1} f_a(n/L_1^{d_f-1})$ and similarly $p'_2(n)$ is of the form $1/L_2^{d_f-1} f_a(n/L_2^{d_f-1})$. The subscript a is added (so $a = e$ or $a = o$) to allow us to distinguish even values of n from odd values, as the DW is topologically very different depending on the parity of n. $p'_1(n \neq 0)$ can mean that macroscopically the DW crosses the central line once (say), but then if one zooms in on that single crossing, it actually crosses many times. To see how many times it would cross, suppose the DW has fractal dimension d_f. Then, the intersection of the DW and a vertical line has dimension $d_f - 1$. Thus, the expected number of crossings would be $L_1^{d_f-1}$ which gives the above scaling forms.

In Fig. 3 we display the data, with 'center' referring to $p'_1(n)$ and 'right' $p'_2(n)$. The scaling functions for even and odd seem to be very similar, even though the DWs are topologically very different. Notice that the no-crossing probabilities are clearly not part of this scaling form (which they cannot be if they are non-zero). However, we suspect in the light of the above that the convergence of $p'_1(0)$ and $p'_2(0)$ to each other might be as slow as $1/L^{d_f-1}$, and as d_f is about 1.27 this could be a very slow convergence rate. We have also studied the case of $s = 0.85$, which corresponds to a still deeper cut by the slits. For this case, the convergence of even p'_1 to p'_2 has not been achieved in the largest sizes we have studied $(146*175)$, and we suspect that the slow convergence here is of similar origin.

We shall now briefly explore the consequences of assuming conformal invariance of the distribution function of the domain walls; we write this distribution function as $P(r_1, r_2, C)$, where r_1, r_2 are the endpoints of the curves and C describes a curve with given endpoints. We have of course provided numerical evidence for only CI of the crossing probabilities, due to limitations on the system sizes that may be studied. We then write $P(r_1, r_2, C) = P(r_1, r_2) * P(C)$, where $P(r_1, r_2)$ is the probability distribution of the endpoints, and $P(C)$ is the conditional probability distirubtion of the curve with the given endpoints. CI of $P(r_1, r_2, C)$ implies CIs of $P(C)$.

Furthermore, we have numerical evidence [13] that DWs exhibit the so-called *Markov Property* [10]. This means basically, if one considers the DWs to be created by a stochastic growth process, then the probability distribution describing the growth at the tip of the DW does not depend on the so-far grown DW. DWs, which exhibit both CI and Markov Property, are described by SLE [10]. The first important consequence is that this implies a relationship between the fractal dimension of the DW d_f and the exponent θ. SLE processes are characterized by a diffusion constant κ. In terms of this parameter, the fractal dimension is determined by κ via

$$d_f = 1 + \frac{\kappa}{8}. \tag{5}$$

The correlation length exponent exponent ν is related to one of the Kac elements of conformal field theory: for example in Potts models with components Q, $1 \leq Q \leq 4$, one has $d - 1/\nu = 2 + \theta = 2h_{2,1}$, but in other models $d - 1/\nu$ is $2h_{1,3}$ or $2h_{1,2}$. d is the dimensionality of the system i.e. 2. Now if SLE applies, each of these elements is related to κ: $2h_{2,1} = (6-\kappa)/\kappa$, $2h_{1,3} = \kappa - 2$ etc. For each of these possibilities one can derive a relationship between d_f and θ and the only one which comes close to the numerical values $d_f = 1.27 \pm 0.01$ and $\theta = -0.285 \pm$

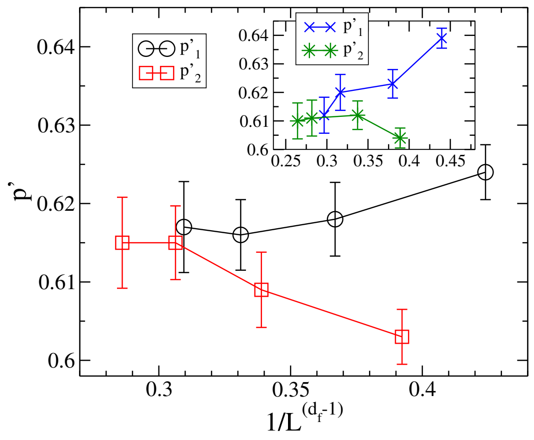

FIG. 2: Approach of p_1' and p_2' to asymptopia. Main figure is for $s = 0.95$ while inset for $s = 0.90$.

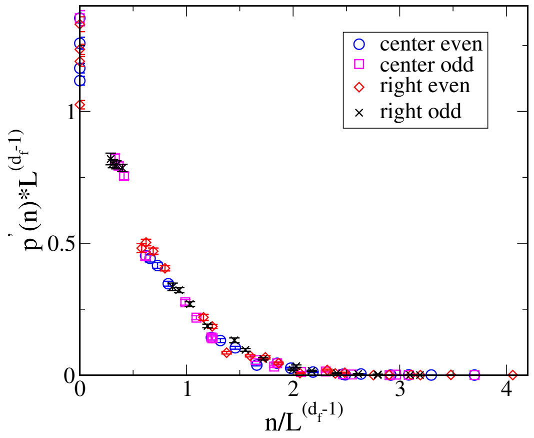

FIG. 3: Scaling of probability of crossings for $s = 0.95$.

0.05 is from $2h_{2,1} = \frac{6-\kappa}{\kappa} = d - \frac{1}{\nu} = d + \theta$. Then on eliminating κ using Eq. (5)

$$d_f = 1 + \frac{3}{4(3+\theta)}. \tag{6}$$

On using one of the alternative possible relations, say $2+\theta = 2h_{1,3}$, $d_f = (12+\theta)/8$. Then the predicted value of $d_f = 1.46$ which is not consistent with its numerical value. Eq. (6) is the only possible relationship between d_f and θ which seems compatible with their numerically well-established values.

To summarize, we have provided direct numerical evidence for CI in an disordered 2d model, namely an Ising spin glass. Together with numerical evidence for the Markov Property [13], this means that DWs in 2d spin glasses are SLE processes, having important implications such as the relation between d_f and θ.

Extensions of this program to other models could be envisaged, in particular the $\pm J$ model. For this model it is important to distinguish between properties defined right at zero temperature and in the limit T going to zero. The exponent θ is thought to be exactly 0 [9]. If Eq. (6) holds for this model too, $d_f = \frac{5}{4}$ and $\kappa = 2$. This would imply that the DWs should be SLE_2 processes, just like loop-erased random walks which are in the universality class of the $O(n)$ model with $n = -2$ [10]. Thus if this connection really holds all the exponents of the $\pm J$ model could be determined from the existing studies of these other models. Recently, $d_f = 1.256(54)$ has been obtained from the scaling of the entropy of non-zero energy DWs[14], which is compatible with Eq. (9), although other results were found recently[15].

We would like to thank Marco Picco for a very informative correspondance as well as Andreas Honecker and Ron Fisch for fruitful discussions. This work was supported from the European Community via the DYGLAGEMEM contract. AKH acknowledges financial support from the *VolkswagenStiftung* (Germany) within the program "Nachwuchsgruppen an Universitäten". MBH was supported by DOE W-7405-ENG-36.

[1] P. Di Francesco, P. Mathieu and D. Senechal *Conformal field theory* (Springer, 1997).
[2] A. Honecker, M. Picco, and P. Pujol, Phys. Rev. Lett. **87**, 047201 (2001).
[3] F. Merz and J.T. Chalker, Phys. Rev. B **65**, 054425 (2002)
[4] H. G. Katzgraber, L. W. Lee and A. P. Young, Phys. Rev. B **70**, 014417 (2004).
[5] A. J. Bray and M. A. Moore, in *Glassy Dynamics and Optimization*, edited by J. L. van Hemmen and I. Morgenstern, (Springer, Berlin, 1986); D. S. Fisher and D. A. Huse, Phys. Rev. Lett. **56**, 1601 (1986); Phys. Rev. B **38**, 386 (1988); W. L. McMillan, J. Phys. C **17**, 3179 (1984).
[6] A. C. Carter, A. J. Bray and M. A. Moore, Phys. Rev. Lett. **88**, 077201 (2002).
[7] A. K. Hartmann, A. J. Bray, A. C. Carter, M. A. Moore and A. P. Young, Phys. Rev. B **66**, 224401 (2002).
[8] A. K. Hartmann and A. P. Young, Phys. Rev. B **66**, 094419 (2002).
[9] A. P. Young and A. K. Hartmann, Phys. Rev. B **64**, 180404 (2001).
[10] For a reviews aimed at physicists, see J. Cardy, Annals of Physics, **318**, 81 (2005); M. Bauer and D. Bernard, preprint math-ph/0602049.
[11] Function $g_h(z)$ is related by a logarithmic map to one given in H. Kober, *Dictionary of Conformal Representations* (Dover 1957), section 13.11, p. 192.
[12] A. K. Hartmann and H. Rieger, *Optimization Algorithms in Physics*, (Wiley-VCH, Berlin 2001).
[13] We have studied numerically the probability distributions for DWs in different models. For percolation, where it is known that the Markov Property holds, and for DWs in 2d spin glasses, the numerical distributions do not show any dependence on the so-far grown domain wall, while they do so for the so-called *boundary harmonic navigator*, where the Markov Property does not hold. Details will be published elsewhere.
[14] R. Fisch, preprint cond-mat/0508468.

[15] R. Fisch, private communication.