LAPLACIAN EIGENVALUES FOR LARGE NEGATIVE ROBIN PARAMETERS ON DOMAINS WITH OUTWARD PEAKS

KONSTANTIN PANKRASHKIN, FIROJ SK, AND MARCO VOGEL

Abstract. We study the asymptotic behavior of individual eigenvalues of Laplacians in domains with outward peaks for large negative Robin parameters. A large class of cross-sections is allowed, and the resulting asymptotic expansions reflect both the sharpness of the peak and the geometric shape of its cross-section. The results are an extension of previous works dealing with peaks whose cross-sections are balls.

1. Introduction

If $\Omega \subset \mathbb{R}^d$ is a bounded domain (non-empty connected open set) with suitably regular boundary, we are interested in the following Robin eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda u \text{ in } \Omega, \\ \partial_\nu u = \alpha u \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\alpha > 0$ is a parameter and ∂_ν means the outer normal derivative. The problem (1) is understood in a weak sense. For a strict formulation we denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure, and for the sake of readability we denote a function in Ω and its Sobolev trace on $\partial\Omega$ by the same symbol. Under appropriate regularity assumptions on Ω the symmetric bilinear form r_α^Ω defined on the domain $\mathcal{D}(r_\alpha^\Omega) = W^{1,2}(\Omega)$ by

$$r_\alpha^\Omega(u,v) := \int_\Omega \langle \nabla u, \nabla v \rangle_{\mathbb{R}^d} \, \mathrm{d}\mathcal{H}^d - \alpha \int_{\partial\Omega} uv \, \mathrm{d}\mathcal{H}^{d-1}$$

is closed and generates a self-adjoint operator R_α^Ω with compact resolvent in $L^2(\Omega)$. Informally, the operator R_α^Ω corresponds to the Laplacian $u \mapsto -\Delta u$ acting on the functions u satisfying the boundary condition $\partial_\nu u = \alpha u$ on $\partial\Omega$, and the eigenvalues λ in (1) are understood as the eigenvalues of R_α^Ω. In particular, a number λ is an eigenvalue of (1) with an eigenfunction u if and only if

$$r_\alpha^\Omega(u,v) = \lambda \int_\Omega uv \, \mathrm{d}\mathcal{H}^d \text{ for all } v \in W^{1,2}(\Omega).$$

Alternatively, one can consider $u \mapsto r_\alpha^\Omega(u,u)$ as the energy functional for (1) and characterize the eigenvalues variationally using the min-max principle. Remark that the boundary term in the above expression for $r_\alpha^\Omega(u,u)$ is negative for $\alpha > 0$, so this case is usually termed as the case of *negative* Robin parameters.

In what follows, for a lower semibounded self-adjoint operator A we denote by $\lambda_j(A)$ its j-th eigenvalue (if it exists), assuming that the eigenvalues are enumerated in the non-decreasing order by taking into account their multiplicities. The goal of the present paper is to study the asymptotic behavior of the individual eigenvalues of R_α^Ω, i.e. of $\lambda_j(R_\alpha^\Omega)$ with fixed j, for $\alpha \to +\infty$ and a special class of domains Ω. The dependence of $\lambda_j(R_\alpha^\Omega)$ on α for various classes of Ω has been given a considerable attention during the last decade, see the reviews in [4, 9]. If

2020 *Mathematics Subject Classification.* 35P15; 35P20; 47A75; 49R05; 58C40.

Key words and phrases. Laplacian, Robin boundary condition, Negative eigenvalues, Asymptotic expansion, Domains with peaks.

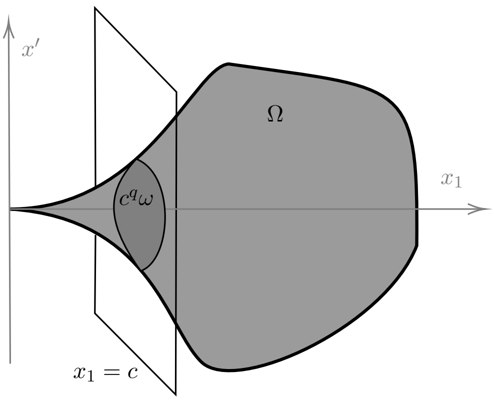

FIGURE 1. An example of a domain Ω with an outward peak with cross-section ω.

Ω is a bounded Lipschitz domain, then all required regularity assumptions are satisfied, and it is known that for large α one has a two-sided bound

$$-c\alpha^2 \leq \lambda_1(R_\alpha^\Omega) \leq -\alpha^2 \tag{2}$$

with some $c \geq 1$, see e.g. [4, Prop. 4.12] and [9, Lem. 2.7]. Under stronger regularity assumptions one can construct detailed asymptotic expansions for $\lambda_j(R_\alpha^\Omega)$ with any fixed j involving various geometric properties of Ω and $\partial\Omega$, see e.g. [2, 3, 6, 7, 8, 11, 14] and the reviews in [4, 9]. On the other hand, it was observed in [11] that (2) fails for domains Ω with outward peaks, which was later studied in greater detail in [10]. As the subsequent text is specifically devoted to the study of $\lambda_j(R_\alpha^\Omega)$ for such Ω, let us introduce an adapted language in order to continue the discussion.

From now on let $d \in \mathbb{N}$ with $d \geq 2$. The vectors $x \in \mathbb{R}^d$ will be written in the form $x = (x_1, x')$ with $x' \in \mathbb{R}^{d-1}$. The following definition is in the spirit of [12, Sec. 5.1.1], see Figure 1 for an illustration:

Definition 1. Let $q > 1$ and $\omega \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain. We say that a bounded domain $\Omega \subset \mathbb{R}^d$ has an outward peak at the origin, of sharpness order q with cross-section ω, if for some $\delta > 0$ it holds

$$\Omega \cap (-\delta, \delta)^d = \Big\{ x \in \mathbb{R}^d : \ x_1 \in (0, \delta), \ x' \in x_1^q \omega \Big\}$$

and Ω is Lipschitz at all boundary points except at the origin.

The main result of the present work is as follows:

Theorem 2. *Let $\Omega \subset \mathbb{R}^d$ be as in Definition 1 with some bounded Lipschitz cross-section $\omega \subset \mathbb{R}^{d-1}$ and sharpness order $q \in (1, 2)$, then for any fixed $j \in \mathbb{N}$ one has*

$$\lambda_j(R_\alpha^\Omega) = \left(\frac{\mathcal{H}^{d-2}(\partial\omega)}{\mathcal{H}^{d-1}(\omega)} \right)^{\frac{2}{2-q}} \lambda_j(L_1) \alpha^{\frac{2}{2-q}} + O\left(\alpha^{\frac{2}{2-q} - (q-1)} \right) \ \text{for } \alpha \to +\infty, \tag{3}$$

where L_1 is the differential operator in $L^2(0, \infty)$ defined first by

$$(L_1 f)(s) := -f''(s) + \left(\frac{q^2(d-1)^2 - 2q(d-1)}{4s^2} - \frac{1}{s^q} \right) f(s), \quad f \in C_c^\infty(0, \infty),$$

and then extended using the Friedrichs extension.

Let us make some comments on the assumptions and the relations with earlier works. All eigenvalues $\lambda_j(L_1)$ are negative, see Subsection 2.2 below, so the eigenvalues $\lambda_j(R_\alpha^\Omega)$ diverge to $-\infty$ much faster than in the Lipschitz case. We are not aware of any value of $q \in (1, 2)$ for which the eigenvalues of L_1 can be computed explicitly. The assumption $q \in (1,2)$ is necessary to ensure that the Sobolev trace operator is well-defined and compact from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$, which guarantees the required properties of the above bilinear form r_α^Ω and the self-adjointness of the Robin Laplacian R_α^Ω for all $\alpha > 0$, see [5, Sec. 5.2] or [1, 13]. The case when ω is a ball of radius $\rho > 0$ was already studied in the earlier paper [10], and its main result is recovered (with an improved remainder estimate) by using (3) and observing that in this case

$$\frac{\mathcal{H}^{d-2}(\partial\omega)}{\mathcal{H}^{d-1}(\omega)} = \frac{d-1}{\rho}.$$

It should be noted that the analysis in [10] was crucially depending on a separation of variables in the ball (and lengthy manipulations with asymptotic expansions of special functions), which is indeed unavailable for general ω. The central observation in this paper is that a part of the arguments of [10] based on a separation of variables can be replaced by an argument based on the first-order perturbation theory of linear operators and on an adapted coordinate change, and it was mainly motivated by observations and geometric constructions from the later paper [15]. The asymptotic expansion (3) reflects both the sharpness q (through the order in α) and the geometry of the cross-section ω (through the coefficient in the main term), which is an essentially new contribution when compared to [10]. This allows to make additional observations on the interplay between the geometry and the eigenvalues. For example, if the area or the perimeter of ω is fixed, then the ratio $\mathcal{H}^{d-2}(\partial\omega)/\mathcal{H}^{d-1}(\omega)$ appearing in the main term of the asymptotic expansion is minimized by the balls due to the isoperimetric inequality. Therefore, if the peak cross-section is a ball, then for $\alpha \to +\infty$ the individual eigenvalues of R_α^Ω diverge to $-\infty$ slower than for any other peak cross-section having the same area or the same perimeter.

The study of Robin eigenvalues can be addressed, in principle, for more general peaks. First, one can extend the class of possible Ω by replacing the condition $x' \in x_1^q\omega$ with the more general one $x' \in \varphi(x_1)\omega$, where φ is a strictly increasing smooth function with $\varphi(0) = \varphi'(0) = 0$. While some first steps of the analysis are still applicable, the absence of principal homogeneous terms in various intermediate operators poses severe problems for the description of the eigenvalue asymptotics, and no power-type asymptotics in α can be expected. A further possible generalization is admitting non-Lipschitz cross-sections ω (in particular, those having peaks in a suitable defined sense). This case of "iterated peaks" is likely to give rise to a multiscale analysis of Laplacians in several dimensions, which is expected to be at a much higher complexity level than the present work. Another extension arises if one admits so-called non-isotropic peaks featuring different scalings in different x'-directions. In this case a multi-step approach seems promising, and a particular case could recently be analyzed in [17].

The overall structure of the paper and of the proof follows closely the one in [10]. Section 2 is devoted to technical preparations. In Subsection 2.1 we collect important facts on Robin Laplacians proved in previous works. In Subsection 2.2 we introduce a family of one-dimensional operators, which includes the operator L_1 appearing in Theorem 2, and recall their basic spectral and asymptotic properties. Subsection 2.3 introduces a family of model peak domains together with associated Laplace-type operators. In Section 2.4 a coordinate change is employed to map the peak domains diffeomorphically to cylindrical domains and to control various integral terms. This part is new with respect to [10], and it is an adaptation of some computations from [15]. In Subsection 2.5, we obtain a lower bound for model operators defined outside a neighborhood of the peak's tip. The core of the study is Section 3, in which we study the eigenvalues of a model operator defined in a small neighborhood of the peak's tip and relate them to the eigenvalues of L_1 as summarized in Corollary 14. The proof avoids using a separation of variables and employs the general results on Robin Laplacians from Section 2.1 instead. In the last section (Section 4) we make use of a series of truncations of Ω to show that only a small neighborhood of the peak's tip counts for the main term of the eigenvalue asymptotics, and an application of Corollary 14 completes the proof of Theorem 2. Most spectral estimates are of purely variational nature and obtained mainly with the help of the min-max principle for eigenvalues through an adapted choice of test functions.

For the rest of the paper we assume that $\Omega \subset \mathbb{R}^d$ satisfies the assumptions of Theorem 2. This fixes once and for all the sharpness parameter $q \in (1, 2)$ and the cross-section $\omega \subset \mathbb{R}^{d-1}$, and we denote additionally

$$A_\omega := \frac{\mathcal{H}^{d-2}(\partial\omega)}{\mathcal{H}^{d-1}(\omega)}.$$

2. Preparations

2.1. **Robin Laplacians on bounded Lipschitz domains.** The following proposition collects important properties of the eigenvalues of the Robin Laplacians, and we refer to [15, Lemma 2.1] for a proof.

Lemma 3. *Let $U \subset \mathbb{R}^m$ be a bounded Lipschitz domain, then the following properties hold true for the Robin Laplacian R^U_α (as defined in the introduction):*

(i) *Scaling: For any $t > 0$, $\alpha \in \mathbb{R}$, $j \in \mathbb{N}$ one has*

$$\lambda_j(R^{tU}_\alpha) = \frac{\lambda_j(R^U_{t\alpha})}{t^2}.$$

(ii) *For any $\alpha \in \mathbb{R}$, the first eigenvalue $\lambda_1(R^U_\alpha)$ is simple and the corresponding eigenfunctions have constant sign in U.*

(iii) *The function*

$$\mathbb{R} \ni \alpha \mapsto \lambda_1(R^U_\alpha) \in \mathbb{R}$$

is smooth. If the associated eigenfunction ψ_α is chosen positive with $\|\psi_\alpha\|_{L^2(U)} = 1$, then the mapping

$$\mathbb{R} \ni \alpha \mapsto \psi_\alpha \in L^2(U)$$

is also smooth.

(iv) *There exists $\phi \in L^\infty(0, \infty)$ such that*

$$\lambda_1(R^U_\alpha) = -\frac{\mathcal{H}^{m-1}(\partial U)}{\mathcal{H}^m(U)}\alpha + \alpha^2\phi(\alpha) \text{ for all } \alpha > 0.$$

(v) *If N_U is the Neumann Laplacian in U, then*

$$\lim_{\alpha\to 0} \lambda_2(R^U_\alpha) = \lambda_2(N_U) > 0.$$

In what follows, Lemma 3 will be used mainly for $m = d - 1$ and $U = \omega$.

2.2. **A one-dimensional operator.** For $\mu > 0$ consider the symmetric differential operator in $L^2(0, \infty)$ given by

$$C^\infty_c(0, \infty) \ni f \mapsto -f'' + \left(\frac{q^2(d-1)^2 - 2q(d-1)}{4s^2} - \frac{\mu}{s^q}\right) f$$

and denote by L_μ its Friedrichs extension. Its essential spectrum is $[0, +\infty)$, and it has infinitely many negative eigenvalues due to the presence of the negative long-range potential μ/s^q, and all the negative eigenvalues are simple. One easily checks that for the unitary scaling transformation

$$Z_c : \ L^2(0, \infty) \to L^2(0, \infty), \quad Z_c f = \sqrt{c} f(c\,\cdot), \quad c > 0,$$

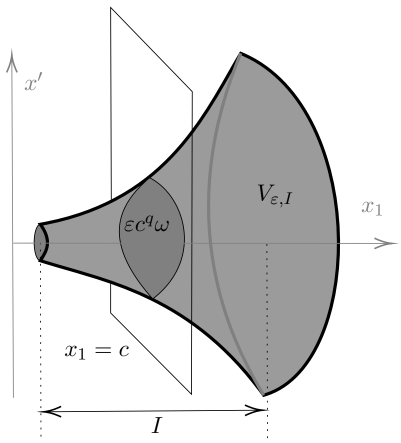

FIGURE 2. An example of the model peak domain $V_{\varepsilon,I}$.

one has

$$L_\mu Z_c = c^2 Z_c L_{c^{q-2}\mu} \text{ for any } \mu > 0 \text{ and } c > 0, \tag{4}$$

which shows that L_μ is unitarily equivalent to $c^2 L_{c^{q-2}\mu}$. For $c := \mu^{\frac{1}{2-q}}$ this implies the identity

$$\lambda_j(L_\mu) = \mu^{\frac{2}{2-q}} \lambda_j(L_1) \text{ for any } \mu > 0 \text{ and } j \in \mathbb{N}.$$

In what follows we will deal with truncated versions of L_μ. Namely, for $a > 0$ we denote by $L_{\mu,a}$ the Friedrichs extension in $L^2(0,a)$ of the operator

$$C_c^\infty(0,a) \ni f \mapsto L_\mu f.$$

By construction the form domain of $L_{\mu,a}$ is continuously embedded in $W_0^{1,2}(0,a)$, which implies that $L_{\mu,a}$ has compact resolvent. In addition, the usual mollification procedure shows that any function from $W^{1,2}(0,a)$ vanishing in some neighborhoods of the endpoints belongs to the form domain of $L_{\mu,a}$ and, moreover, such functions build a core domain for the bilinear form of $L_{\mu,a}$. We will use the following asymptotic estimate for the eigenvalues of $L_{\mu,a}$:

Lemma 4. *For any $a > 0$ and $j \in \mathbb{N}$ there exist $K > 0$ and $\mu_0 > 0$ such that*

$$\mu^{\frac{2}{2-q}} \lambda_j(L_1) \le \lambda_j(L_{\mu,a}) \le \mu^{\frac{2}{2-q}} \lambda_j(L_1) + K \quad \text{for all } \mu > \mu_0.$$

The proof is given in [10, Sec. 3.1] for a slightly different choice of parameters, and it is translated into our language using the scaling (4).

2.3. **Finite peaks and related operators.** For $\varepsilon > 0$ we will consider various finite pieces of the infinite peak

$$V_\varepsilon := \Big\{ (x_1, x') \in \mathbb{R}^d : \ x_1 \in (0,\infty), \ x' \in \varepsilon x_1^q \omega \Big\}.$$

Namely, for an open interval $I \subset (0,\infty)$ it will be convenient to denote

$$V_{\varepsilon,I} := V_\varepsilon \cap \Big(I \times \mathbb{R}^{d-1} \Big) \equiv \Big\{ (x_1, x') \in \mathbb{R}^d : \ x_1 \in I, \ x' \in \varepsilon x_1^q \omega \Big\},$$

see Figure 2. In particular, one has $V_\varepsilon = V_{\varepsilon,(0,\infty)}$. We will also consider the "lateral boundary" $\partial_0 V_{\varepsilon,I}$ of $V_{\varepsilon,I}$ given by

$$\partial_0 V_{\varepsilon,I} := \partial V_\varepsilon \cap \Big(I \times \mathbb{R}^{d-1} \Big) \equiv \Big\{ (x_1, x') \in \mathbb{R}^d : \ x_1 \in I, \ x' \in \varepsilon x_1^q \partial\omega \Big\}.$$

At several places we will work with functions localized in the first variable, so let us introduce an adapted notation. For an open set $U \subset \mathbb{R}^d$ and an open interval $I \subset \mathbb{R}$ it will be convenient to denote

$$W^{1,2}_I(U) := \Big\{ u \in W^{1,2}(U) :\ \exists c, c' \in I \text{ such that } u(x) = 0 \text{ for all } x \in U \text{ with } x_1 \notin [c, c'] \Big\}.$$

For $\alpha \in \mathbb{R}$ and an open interval $I \subset (0,\infty)$ consider the symmetric bilinear form $t^{\alpha,N}_{\varepsilon,I}$ given by

$$t^{\alpha,N}_{\varepsilon,I}(u,u) := \int_{V_{\varepsilon,I}} |\nabla u|^2 \mathrm{d}\mathcal{H}^d - \alpha \int_{\partial_0 V_{\varepsilon,I}} u^2 \mathrm{d}\mathcal{H}^{d-1}, \qquad \mathcal{D}(t^{\alpha,N}_{\varepsilon,I}) := W^{1,2}(V_{\varepsilon,I}).$$

In the final steps of the proof of the main theorem we will use the following density result:

Lemma 5. *For any open bounded interval $I \subset (0,\infty)$ and any $\varepsilon > 0$ the set $W^{1,2}_{(0,\infty)}(V_{\varepsilon,I})$ is dense in $W^{1,2}(V_{\varepsilon,I})$.*

Proof. Remark that $W^{1,2}_{(0,\infty)}(V_{\varepsilon,I})$ consists of the functions in $W^{1,2}(V_{\varepsilon,I})$ that vanish in a neighborhood of the origin. If 0 is not an endpoint of I, then $W^{1,2}_{(0,\infty)}(V_{\varepsilon,I}) = W^{1,2}(V_{\varepsilon,I})$, and there is nothing to prove. Therefore, from now on we consider the case $I = (0,\ell)$ with $\ell \in (0,\infty)$.

From the general theory of Sobolev spaces it is known that $W^{1,2}(V_{\varepsilon,I}) \cap L^\infty(V_{\varepsilon,I})$ is a dense subspace of $W^{1,2}(V_{\varepsilon,I})$, see [12, Theorem in Sec. 1.4.3]. Therefore, it is sufficient to show that any $u \in W^{1,2}(V_{\varepsilon,I}) \cap L^\infty(V_{\varepsilon,I})$ can be approximated by functions from $W^{1,2}_{(0,\infty)}(V_{\varepsilon,I})$. Pick a function $\chi \in C^\infty(0,\infty)$ such that

$$0 \le \chi \le 1, \qquad \chi(s) = 0 \text{ for } s < 1, \qquad \chi(s) = 1 \text{ for } s > 2,$$

and for small $\mu > 0$ consider the functions

$$u_\mu :\ (x_1, x') \mapsto \chi\Big(\frac{x_1}{\mu}\Big) u(x_1, x').$$

By construction $u_\mu \in W^{1,2}(V_{\varepsilon,I})$ with $u_\mu(x_1,x') = 0$ for $x_1 < \mu$, so $u_\mu \in W^{1,2}_{(0,\infty)}(V_{\varepsilon,I})$. We are going to show that u_μ converges to u in $W^{1,2}(V_{\varepsilon,I})$ for $\mu \to 0^+$.

Using the dominated convergence theorem one shows that u_μ converges to u in $L^2(V_{\varepsilon,I})$ for $\mu \to 0^+$. For each $j \ge 2$ we have

$$\partial_j u_\mu :\ (x_1, x') \mapsto \chi\Big(\frac{x_1}{\mu}\Big) \partial_j u(x_1, x'),$$

and the same argument shows the convergence of $\partial_j u_\mu$ to $\partial_j u$ in $L^2(V_{\varepsilon,I})$ for $\mu \to 0^+$. Furthermore,

$$\partial_1 u_\mu :\ (x_1, x') \mapsto \frac{1}{\mu}\chi'\Big(\frac{x_1}{\mu}\Big) u(x_1, x') + \chi\Big(\frac{x_1}{\mu}\Big) \partial_1 u(x_1, x'),$$

in particular, $\partial_1 u_\mu(x_1,x') = \partial_1 u(x_1,x')$ for $x_1 > 2\mu$, and

$$\begin{aligned}
\|\partial_1 u_\mu - \partial_1 u\|^2_{L^2(V_{\varepsilon,I})} &= \|\partial_1 u_\mu - \partial_1 u\|^2_{L^2(V_{\varepsilon,(0,2\mu)})} \\
&= \int_0^{2\mu} \int_{\varepsilon x_1^q \omega} \bigg(\frac{1}{\mu}\chi'\Big(\frac{x_1}{\mu}\Big) u(x_1,x') + \Big[\chi\Big(\frac{x_1}{\mu}\Big) - 1\Big] \partial_1 u(x_1,x') \bigg)^2 \mathrm{d}\mathcal{H}^{d-1}(x')\, \mathrm{d}x_1 \\
&\le I'_\mu + I''_\mu, \\
I'_\mu &:= \frac{2}{\mu^2} \int_0^{2\mu} \int_{\varepsilon x_1^q \omega} \Big[\chi'\Big(\frac{x_1}{\mu}\Big) u(x_1,x') \Big]^2 \mathrm{d}\mathcal{H}^{d-1}(x')\, \mathrm{d}x_1, \\
I''_\mu &:= 2 \int_0^{2\mu} \int_{\varepsilon x_1^q \omega} \bigg(\Big[\chi\Big(\frac{x_1}{\mu}\Big) - 1\Big] \partial_1 u(x_1,x') \bigg)^2 \mathrm{d}\mathcal{H}^{d-1}(x')\, \mathrm{d}x_1.
\end{aligned}$$

Using $q(d-1)>1$ we obtain

$$
\begin{aligned}
I'_\mu &\le \frac{2}{\mu^2}\int_0^{2\mu}\int_{\varepsilon x_1^q\omega}\|\chi'\|_\infty^2\|u\|_\infty^2 \mathrm{d}\mathcal{H}^{d-1}(x')\,\mathrm{d}x_1\\
&= \frac{2\|\chi'\|_\infty^2\|u\|_\infty^2}{\mu^2}\int_0^{2\mu}\mathcal{H}^{d-1}(\varepsilon x_1^q\omega)\,\mathrm{d}x_1\\
&= \frac{2\|\chi'\|_\infty^2\|u\|_\infty^2}{\mu^2}\int_0^{2\mu}(\varepsilon x_1^q)^{d-1}\mathcal{H}^{d-1}(\omega)\,\mathrm{d}x_1\\
&= \frac{2\|\chi'\|_\infty^2\|u\|_\infty^2\varepsilon^{d-1}\mathcal{H}^{d-1}(\omega)}{\mu^2}\int_0^{2\mu}x_1^{q(d-1)}\mathrm{d}x_1\\
&= \frac{2\|\chi'\|_\infty^2\|u\|_\infty^2\varepsilon^{d-1}\mathcal{H}^{d-1}(\omega)}{\mu^2}\,\frac{(2\mu)^{q(d-1)+1}}{q(d-1)+1}\\
&= \frac{2^{q(d-1)+2}\|\chi'\|_\infty^2\|u\|_\infty^2\varepsilon^{d-1}\mathcal{H}^{d-1}(\omega)}{q(d-1)+1}\mu^{q(d-1)-1}\xrightarrow{\mu\to0^+}0,
\end{aligned}
$$

while I''_μ converges to 0 for $\mu\to0^+$ due to the dominated convergence theorem, which implies the convergence of $\partial_1 u_\mu$ to $\partial_1 u$ in $L^2(V_{\varepsilon,I})$ for $\mu\to0^+$. □

Denote

$$\widehat{W}_0^{1,2}(V_{\varepsilon,I}) := \text{the closure of } W_I^{1,2}(V_{\varepsilon,I}) \text{ in } W^{1,2}(V_{\varepsilon,I})$$

and consider the symmetric bilinear form

$$t_{\varepsilon,I}^{\alpha,D} := \text{the restriction of } t_{\varepsilon,I}^{\alpha,N} \text{ on } \widehat{W}_0^{1,2}(V_{\varepsilon,I}).$$

By construction we have:

$$W_I^{1,2}(V_{\varepsilon,I}) \text{ is a core domain of } t_{\varepsilon,I}^{\alpha,D}, \tag{5}$$

which will simplify the subsequent considerations.

In what follows we will be interested in the spectral analysis of the self-adjoint operators $T_{\varepsilon,I}^{\alpha,N/D}$ acting in $L^2(V_{\varepsilon,I})$ and defined by the forms $t_{\varepsilon,I}^{\alpha,N/D}$. Note that for any $\alpha>0$ and any I one has

$$
\begin{aligned}
\alpha V_{1,I} &= \Big\{(x_1,x')\in\mathbb{R}^d:\ (\frac{x_1}{\alpha},\frac{x'}{\alpha})\in V_{1,I}\Big\}\\
&= \Big\{(x_1,x')\in\mathbb{R}^d:\ \frac{x_1}{\alpha}\in I,\ \frac{x'}{\alpha}\in(\frac{x_1}{\alpha})^q\omega\Big\}\\
&= \Big\{(x_1,x')\in\mathbb{R}^d:\ x_1\in\alpha I,\ x'\in\alpha^{1-q}x_1^q\omega\Big\}\equiv V_{\alpha^{1-q},\alpha I}.
\end{aligned}
$$

Therefore, a simple scaling argument shows that for any $j\in\mathbb{N}$ one has the relations

$$\lambda_j(T_{1,I}^{\alpha,N/D}) = \alpha^2\lambda_j(T_{\alpha^{1-q},\alpha I}^{1,N/D}). \tag{6}$$

Our next goal is to obtain lower and upper bounds for the eigenvalues of $T_{\varepsilon,I}^{\alpha,N/D}$ with the help of suitable coordinate changes.

2.4. **Reduction to cylindrical domains.** For $\varepsilon>0$ consider the diffeomorphism

$$F_\varepsilon:(0,\infty)\times\mathbb{R}^{d-1}\to(0,\infty)\times\mathbb{R}^{d-1},\qquad F_\varepsilon(s,t):=(s,\varepsilon s^q t),$$

then for any open interval $I\subset(0,\infty)$ we have

$$F_\varepsilon(\Pi_I)=V_{\varepsilon,I}\quad\text{for}\quad \Pi_I:=I\times\omega.$$

In order to deal with various integrals under the change of variables defined by F_ε, we will perform some preliminary computations. Let us introduce the constant

$$R_\omega := \sup_{t\in\omega} |t|.$$

Lemma 6. *For any $\varepsilon > 0$, any measurable function $v : \partial_0 V_{\varepsilon,I} \to \mathbb{R}$ and $u := v \circ F_\varepsilon$ it holds*

$$\varepsilon^{d-2} \int_I s^{q(d-2)} \int_{\partial\omega} |u(s,t)|\, \mathrm{d}\mathcal{H}^{d-2}(t)\, \mathrm{d}s \le \int_{\partial_0 V_{\varepsilon,I}} |v|\, \mathrm{d}\mathcal{H}^{d-1}$$
$$\le \varepsilon^{d-2} \int_I s^{q(d-2)} \sqrt{1+\varepsilon^2 q^2 R_\omega^2\, s^{2q-2}} \int_{\partial\omega} |u(s,t)|\, \mathrm{d}\mathcal{H}^{d-2}(t)\, \mathrm{d}s.$$

Proof. It is sufficient to prove the result for the functions v supported in images of local charts, then it is extended to general functions by using a partition of unity. Let

$$U \ni z = (z_1, z_2, \cdots, z_{d-2}) \mapsto \psi(z) \in \partial\omega$$

be a local chart on $\partial\omega$, then $U \ni z \mapsto \varepsilon\psi(z) \in \partial(\varepsilon\omega)$ is a local chart on $\partial(\varepsilon\omega)$, and

$$\Psi_\varepsilon :\ I \times U \ni (s,z) \mapsto \big(s, \varepsilon s^q \psi(z)\big) \equiv F_\varepsilon\big(s,\psi(z)\big) \in \partial_0 V_{\varepsilon,I}$$

is a local chart on $\partial_0 V_{\varepsilon,I}$. If v is supported in the image of Ψ_ε, then the expression of $\mathcal{H}^{d-1}$ on hypersurfaces gives

$$\int_{\partial_0 V_{\varepsilon,I}} |v|\, \mathrm{d}\mathcal{H}^{d-1} = \int_I \int_U \Big|v\big(\Psi_\varepsilon(s,z)\big)\Big| J_\varepsilon(s,z)\, \mathrm{d}\mathcal{H}^{d-2}(z)\mathrm{d}s \tag{7}$$

with

$$J_\varepsilon(s,z) := \sqrt{\det\big(D\Psi_\varepsilon(s,z)^T D\Psi_\varepsilon(s,z)\big)}.$$

A direct computation gives

$$D\Psi_\varepsilon(s,z) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ \varepsilon q s^{q-1}\psi(z) & \varepsilon s^q \partial_1\psi(z) & \dots & \varepsilon s^q \partial_{d-2}\psi(z) \end{pmatrix},$$

which yields

$$D\Psi_\varepsilon(s,z)^T D\Psi_\varepsilon(s,z) = \begin{pmatrix} 1+\varepsilon^2 q^2 s^{2q-2}|\psi(z)|^2 & \varepsilon^2 q s^{2q-1} H(z)^T \\ \\ \varepsilon^2 q s^{2q-1} H(z) & \varepsilon^2 s^{2q} G(z) \end{pmatrix}$$

with $G(z) := D\psi(z)^T D\psi(z)$ and

$$H(z) := \begin{pmatrix} \langle \psi(z), \partial_1\psi(z)\rangle \\ \vdots \\ \langle \psi(z), \partial_{d-2}\psi(z)\rangle \end{pmatrix} \equiv \Big(|\psi|\nabla|\psi|\Big)(z).$$

Since ψ is a local chart, the matrix $G(z)$ is invertible for a.e. z, and a standard formula for computing the determinant of a block matrix with an invertible diagonal block becomes applicable, which leads to

$$J_\varepsilon(s,z)^2 = \varepsilon^{2(d-2)} s^{2q(d-2)} \det G(z)\Big[1+\varepsilon^2 q^2 s^{2q-2}|\psi(z)|^2 - \varepsilon^2 q^2 s^{2q-2} H(z)^T G(z)^{-1} H(z)\Big]$$
$$= \varepsilon^{2(d-2)} s^{2q(d-2)} \det G(z)\Big[1+\varepsilon^2 q^2 s^{2q-2}|\psi(z)|^2 \Big(1 - \Big\langle \nabla|\psi|(z), G(z)^{-1}\nabla|\psi|(z)\Big\rangle\Big)\Big].$$

Consider the modulus function

$$h :\ \partial\omega \ni t \mapsto |t| \in \mathbb{R},$$

then

$$\Big\langle \nabla|\psi|(z), G(z)^{-1}\nabla|\psi|(z)\Big\rangle = |\nabla^{\partial\omega} h|^2\big(\psi(z)\big),$$
$$\nabla^{\partial\omega} h := \text{the tangential gradient of } h \text{ along } \partial\omega,$$

and

$$1 - \Big\langle \nabla|\psi|(z), G(z)^{-1}\nabla|\psi|(z)\Big\rangle = |\nabla h|^2\big(\psi(z)\big) - |\nabla^{\partial\omega} h|^2\big(\psi(z)\big) = |\partial_\nu h|^2\big(\psi(z)\big),$$

where ∂_ν stands for the normal derivative on $\partial\omega$. Therefore,

$$J_\varepsilon(s,z)^2 = \varepsilon^{2(d-2)} s^{2q(d-2)} \det G(z)\Big[1 + \varepsilon^2 q^2 s^{2q-2} \big|\psi(z)\big|^2 |\partial_\nu h|^2\big(\psi(z)\big)\Big],$$

and one arrives at the obvious lower bound

$$J_\varepsilon(s,z)^2 \geq \varepsilon^{2(d-2)} s^{2q(d-2)} \det G(z).$$

Furthermore, due to the definition of h we have $|\partial_\nu h| \leq 1$ a.e, which implies

$$\begin{aligned} J_\varepsilon(s,z)^2 &\leq \varepsilon^{2(d-2)} s^{2q(d-2)} \det G(z)\Big(1 + \varepsilon^2 q^2 s^{2q-2} \big|\psi(z)\big|^2\Big) \\ &\leq \varepsilon^{2(d-2)} s^{2q(d-2)} \det G(z)\Big(1 + \varepsilon^2 q^2 s^{2q-2} R_\omega^2\Big). \end{aligned}$$

By substituting these bounds for $J_\varepsilon(s,z)^2$ into (7) we arrive at

$$\begin{aligned} &\varepsilon^{d-2} \int_I \int_U s^{q(d-2)} \Big|v\big(\Psi_\epsilon(s,z)\big)\Big| \sqrt{\det G(z)}\, \mathrm{d}\mathcal{H}^{d-2}(z)\, \mathrm{d}s \leq \int_{\partial_0 V_{\varepsilon,I}} |v| \mathrm{d}\mathcal{H}^{d-1} \\ &\leq \varepsilon^{d-2} \int_I \int_U s^{q(d-2)} \sqrt{1 + \varepsilon^2 q^2 s^{2q-2} R_\omega^2} \Big|v\big(\Psi_\epsilon(s,z)\big)\Big| \sqrt{\det G(z)}\, \mathrm{d}\mathcal{H}^{d-2}(z)\, \mathrm{d}s. \end{aligned}$$

It remains to note that $v\big(\Psi_\epsilon(s,z)\big) = u\big(s,\psi(z)\big)$ and

$$\int_U \big|u\big(s,\psi(z)\big)\big| \sqrt{\det G(z)}\, \mathrm{d}\mathcal{H}^{d-2}(z) = \int_{\partial\omega} |u(s,\cdot)| \mathrm{d}\mathcal{H}^{d-2}.$$ □

Lemma 7. *For any $v \in W^{1,2}(V_{\varepsilon,I})$ and $u := v \circ F_\varepsilon$ we have*

$$\begin{aligned} \varepsilon^{d-1} \int_I s^{q(d-1)} \int_\omega \bigg[&(1 - (d-1)\varepsilon q R_\omega)|\partial_s u|^2 \\ &+ \left(\frac{1}{\varepsilon^2 s^{2q}} - \frac{qR_\omega}{s^2\varepsilon} - (d-1)\frac{q^2 R_\omega^2}{s^2}\right) |\nabla_t u|^2\bigg] \mathrm{d}\mathcal{H}^{d-1}(t)\, \mathrm{d}s \\ &\leq \int_{V_{\varepsilon,I}} |\nabla v|^2 \mathrm{d}\mathcal{H}^d \\ &\leq \varepsilon^{d-1} \int_I s^{q(d-1)} \int_\omega \bigg[(1 + (d-1)\varepsilon q R_\omega)\, |\partial_s u|^2 \\ &\quad + \left(\frac{1}{\varepsilon^2 s^{2q}} + \frac{qR_\omega}{s^2\varepsilon} + (d-1)\frac{q^2 R_\omega^2}{s^2}\right) |\nabla_t u|^2\bigg] \mathrm{d}\mathcal{H}^{d-1}(t)\, \mathrm{d}s. \end{aligned}$$

Proof. Remark that

$$DF_\varepsilon(s,t) = \begin{pmatrix} 1 & 0 \\ \varepsilon q s^{q-1} t & \varepsilon s^q I_{d-1} \end{pmatrix}, \quad t = \begin{pmatrix} t_1 \\ \dots \\ t_{d-1} \end{pmatrix},$$
$$I_{d-1} := \text{the } (d-1)\times(d-1) \text{ identity matrix}.$$

The change of variables $x = F_\varepsilon(s,t)$ gives

$$\int_{V_{\varepsilon,I}} \big|\nabla v(x)\big|^2 \mathrm{d}\mathcal{H}^d(x) = \int_I \int_\omega \langle \nabla u, G_\varepsilon \nabla u\rangle_{\mathbb{R}^d}\, g_\varepsilon \,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s,$$

where

$$\begin{aligned}
g_\varepsilon(s,t) &:= \Big|\det DF_\varepsilon(s,z)\Big| \equiv \varepsilon^{d-1} s^{q(d-1)},\\
G_\varepsilon(s,t) &:= \Big(DF_\varepsilon(s,t)^T DF_\varepsilon(s,t)\Big)^{-1}\\
&= \begin{pmatrix} 1+\varepsilon^2 q^2\, s^{2(q-1)}|t|^2 & \varepsilon^2 q s^{2q-1} t^T\\ \varepsilon^2 q s^{2q-1} t & \varepsilon^2 s^{2q} I_{d-1}\end{pmatrix}^{-1} = \begin{pmatrix} 1 & -\dfrac{q}{s} t^T\\ -\dfrac{q}{s} t & \dfrac{1}{\varepsilon^2 s^{2q}} I_{d-1} + \dfrac{q^2}{s^2}\, t t^T\end{pmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\langle \nabla u, G_\varepsilon \nabla u\rangle_{\mathbb{R}^d} &= |\partial_s u|^2 + \frac{1}{\varepsilon^2 s^{2q}} |\nabla_t u|^2\\
&\quad - \frac{2q}{s}\sum_{j=1}^{d-1} t_j \partial_s u\,\partial_{t_j} u + \frac{q^2}{s^2}\sum_{j,\,k=1}^{d-1} t_j t_k \partial_{t_j} u\,\partial_{t_k} u.
\end{aligned}\tag{8}$$

We estimate

$$\begin{aligned}
\left|\frac{2q}{s}\sum_{j=1}^{d-1} t_j \partial_s u\,\partial_{t_j} u\right| &\le q\sum_{j=1}^{d-1} |t_j|\, 2\Big|\partial_s u\,\frac{\partial_{t_j}}{s} u\Big| \le qR_\omega \sum_{j=1}^{d-1} 2\Big|\partial_s u\,\frac{\partial_{t_j} u}{s}\Big|\\
&\le qR_\omega \sum_{j=1}^{d-1}\left[\varepsilon|\partial_s u|^2 + \frac{1}{\varepsilon}\Big|\frac{\partial_{t_j} u}{s}\Big|^2\right] = (d-1)qR_\omega\varepsilon|\partial_s u|^2 + \frac{qR_\omega}{\varepsilon s^2}|\nabla_t u|^2.
\end{aligned}\tag{9}$$

Similarly, we have

$$\begin{aligned}
\left|\frac{q^2}{s^2}\sum_{j,\,k=1}^{d-1} t_j t_k \partial_{t_j} u\,\partial_{t_k} u\right| &\le \frac{q^2}{s^2}\sum_{j,\,k=1}^{d-1} |t_j|\,|t_k|\,|\partial_{t_j} u|\,|\partial_{t_k} u|\\
&\le \frac{q^2 R_\omega^2}{s^2}\sum_{j,\,k=1}^{d-1} |\partial_{t_j} u|\,|\partial_{t_k} u|\\
&\le \frac{q^2 R_\omega^2}{2s^2}\sum_{j,\,k=1}^{d-1} \Big(|\partial_{t_j} u|^2 + |\partial_{t_k} u|^2\Big)\\
&= \frac{q^2 R_\omega^2}{s^2}(d-1)|\nabla_t u|^2.
\end{aligned}\tag{10}$$

By using (9) and (10) to estimate the two last summands on the right-hand side of (8) we obtain the desired inequality. □

Note that the above computations are classical for smooth $\partial\omega$, but the formulas are still valid for Lipschitz ω due to general results on Lipschitz manifolds [16].

2.5. **A lower bound outside the peak.** The goal of this subsection is to obtain a lower bound for the eigenvalues of $T^{1,N}_{\varepsilon,I}$ for intervals I separated from 0: It will be used in the truncation arguments in the next sections.

Lemma 8. *Let $b, B > 0$ and $\varepsilon_0 > 0$. Then there exists a constant $c > 0$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and any non-empty open interval*

$$I_\varepsilon \subset \big(b, B\varepsilon^{-\frac{1}{q-1}}\big)$$

there holds $\lambda_1(T^{1,N}_{\varepsilon,I_\varepsilon}) \geq -c\varepsilon^{-1}$.

Proof. For any $u \in \mathcal{D}(T^{1,N}_{\varepsilon,I_\varepsilon})$ we have by Fubini's theorem

$$\int_{V_{\varepsilon,I_\varepsilon}} |\nabla u|^2 \mathrm{d}\mathcal{H}^d \geq \int_{V_{\varepsilon,I_\varepsilon}} |\nabla_{x'} u|^2 \mathrm{d}\mathcal{H}^d = \int_{I_\varepsilon} \int_{\varepsilon x_1^q \omega} \big|\nabla_{x'} u(x_1,x')\big|^2 \mathrm{d}\mathcal{H}^{d-1}(x')\,\mathrm{d}x_1.$$

In addition, due to Lemma 6,

$$\begin{aligned}\int_{\partial_0 V_{\varepsilon,I_\varepsilon}} u^2 \mathrm{d}\mathcal{H}^{d-1} &\leq \varepsilon^{d-2} \int_{I_\varepsilon} x_1^{q(d-2)} \sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}} \int_{\partial\omega} u(x_1, \varepsilon x_1^q x')^2\, \mathrm{d}\mathcal{H}^{d-2}(x')\,\mathrm{d}x_1 \\ &\leq \int_{I_\varepsilon} \sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}} \int_{\partial(\varepsilon x_1^q \omega)} u(x_1,x')^2\, \mathrm{d}\mathcal{H}^{d-2}(x')\,\mathrm{d}x_1.\end{aligned}$$

Therefore,

$$\begin{aligned} t^{1,N}_{\varepsilon,I_\varepsilon}(u,u) &= \int_{V_{\varepsilon,I_\varepsilon}} |\nabla u|^2 \mathrm{d}\mathcal{H}^d - \int_{\partial_0 V_{\varepsilon,I_\varepsilon}} u^2 \mathrm{d}\mathcal{H}^{d-1} \\ &\geq \int_{I_\varepsilon} \Big[\int_{\varepsilon x_1^q \omega} \big|\nabla_{x'} u(x_1,x')\big|^2 \mathrm{d}\,\mathcal{H}^{d-1}(x') \\ &\qquad - \sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}} \int_{\partial(\varepsilon x_1^q\omega)} u(x_1,x')^2\, \mathrm{d}\mathcal{H}^{d-2}(x')\Big]\,\mathrm{d}x_1 \\ &= \int_{I_\varepsilon} r^{\varepsilon x_1^q \omega}_{\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}}}\big(u(x_1,\cdot), u(x_1,\cdot)\big)\,\mathrm{d}x_1 \\ &\geq \int_{I_\varepsilon} \lambda_1\Big(R^{\varepsilon x_1^q \omega}_{\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}}}\Big) \int_{\varepsilon x_1^q \omega} u(x_1,x')^2\, \mathrm{d}\mathcal{H}^{d-1}(x')\,\mathrm{d}x_1 \\ &\geq \Lambda_\varepsilon \int_{V_{\varepsilon,I_\varepsilon}} u^2 \mathrm{d}\mathcal{H}^d, \qquad \Lambda_\varepsilon := \inf_{x_1\in I_\varepsilon} \lambda_1\left(R^{\varepsilon x_1^q \omega}_{\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}}}\right),\end{aligned}$$

i.e. $\lambda_1(T^{1,N}_{\varepsilon,I_\varepsilon}) \geq \Lambda_\varepsilon$ for all $\varepsilon \in (0,\varepsilon_0)$, and it remains to find a suitable lower bound for Λ_ε. With the help of the assertions (i) and (iv) of Lemma 3 we obtain, with some $C' > 0$,

$$\begin{aligned}\lambda_1\left(R^{\varepsilon x_1^q \omega}_{\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}}}\right) &= \frac{\lambda_1\left(R^{\omega}_{\varepsilon x_1^q\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}}}\right)}{\varepsilon^2 x_1^{2q}} \\ &\geq -\frac{\varepsilon A_\omega x_1^q \sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}} + C'\varepsilon^2 x_1^{2q}(1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2})}{\varepsilon^2 x_1^{2q}} \\ &= -A_\omega \frac{\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}}}{\varepsilon x_1^q} - C'(1+\varepsilon^2 q^2\, R_\omega^2\, x_1^{2q-2}).\end{aligned}$$

For all $x_1 \in I_\varepsilon$ we have

$$x_1^q > b^q, \qquad \varepsilon x_1^{q-1} < \varepsilon\big(B\varepsilon^{-\frac{1}{q-1}}\big)^{q-1} = B^{q-1},$$

therefore, for any $\varepsilon \in (0,\varepsilon_0)$ one has

$$\begin{aligned}\Lambda_\varepsilon &\geq -\frac{A_\omega \sqrt{1+q^2 R_\omega B^{2q-2}}}{b^q}\cdot\frac{1}{\varepsilon} - C'(1+q^2 R_\omega^2 B^{2q-2}) \\ &\geq -\Big[\frac{A_\omega\sqrt{1+q^2 R_\omega B^{2q-2}}}{b^q} + C'(1+q^2 R_\omega^2 B^{2q-2})\varepsilon_0\Big]\cdot\frac{1}{\varepsilon}.\end{aligned}$$ $\square$

3. Spectral asymptotics near the peak

3.1. **Model operators near the peak.** We are going to apply a series of coordinate changes, which will allow for the spectral study of the operators

$$T_{\varepsilon,a} := T^{1,D}_{\varepsilon,(0,a)} \tag{11}$$

as $\varepsilon \to 0^+$. We also denote

$$t_{\varepsilon,a} := t^{1,D}_{\varepsilon,(0,a)}, \qquad \Pi_a := \Pi_{(0,a)} \equiv (0,a)\times\omega.$$

The diffeomorphism F_ε defined in Subsection 2.4 induces the unitary transformation

$$\Phi_\varepsilon : L^2(V_{\varepsilon,(0,a)}) \to L^2\big(\Pi_a, \varepsilon^{d-1}s^{(d-1)q}\mathrm{d}s\,\mathrm{d}\mathcal{H}^{d-1}(t)\big),$$
$$(\Phi_\varepsilon u)(s,t) := u\big(F_\varepsilon(s,t)\big).$$

Consider the symmetric bilinear form $p_{\varepsilon,a}$ in $L^2\big(\Pi_a, \varepsilon^{d-1}s^{(d-1)q}\mathrm{d}s\,\mathrm{d}\mathcal{H}^{d-1}(t)\big)$ defined by

$$p_{\varepsilon,a}(u,u) := t_{\varepsilon,a}(\Phi_\varepsilon^{-1}u, \Phi_\varepsilon^{-1}u),$$

on the domain $\mathcal{D}(p_{\varepsilon,a}) = \Phi_\varepsilon\big(\mathcal{D}(t_{\varepsilon,a})\big)$ and the associated self-adjoint operator $P_{\varepsilon,a}$ in the weighted space $L^2\big(\Pi_a, \varepsilon^{d-1}s^{(d-1)q}\mathrm{d}s\,\mathrm{d}\mathcal{H}^{d-1}(t)\big)$, which is by construction unitary equivalent to $T_{\varepsilon,a}$ and, hence, has the same eigenvalues. Remark that due to the explicit form of F_ε and Φ_ε we have $\Phi_\varepsilon\big(W^{1,2}_I(V_{\varepsilon,I})\big) = W^{1,2}_I(\Pi_I)$. It will convenient to denote

$$\mathcal{D}_a := W^{1,2}_{(0,a)}(\Pi_a),$$

then (5) shows that

$$\mathcal{D}_a \text{ is a core domain of } p_{\varepsilon,a},$$

so $\mathcal{D}_a$ can be used as a test domain when applying the min-max principle to $P_{\varepsilon,a}$.

Due to Lemma 6 and Lemma 7 for any $u \in \mathcal{D}_a$ one has

$$p^-_{\varepsilon,a}(u,u) \le p_{\varepsilon,a}(u,u) \le p^+_{\varepsilon,a}(u,u)$$

with symmetric bilinear forms $p^\pm_{\varepsilon,a}(u,u)$ in $L^2\big(\Pi_a, \varepsilon^{d-1}s^{(d-1)q}\mathrm{d}s\,\mathrm{d}\mathcal{H}^{d-1}(t)\big)$ defined on $\mathcal{D}_a$

$$\begin{aligned}
p^-_{\varepsilon,a}(u,u) := \varepsilon^{d-1}\int_0^a s^{q(d-1)}\int_\omega\bigg[&\big(1-(d-1)\varepsilon q R_\omega\big)|\partial_s u|^2\\
&+\bigg(\frac{1}{\varepsilon^2 s^{2q}} - \frac{qR_\omega}{s^2\varepsilon} - (d-1)\frac{q^2R_\omega^2}{s^2}\bigg)|\nabla_t u|^2\bigg]\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s\\
&-\varepsilon^{d-2}\int_0^a s^{q(d-2)}\sqrt{1+\varepsilon^2 q^2\,R_\omega^2\,s^{2q-2}}\int_{\partial\omega}\big|u(s,t)\big|^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s,\\
p^+_{\varepsilon,a}(u,u) := \varepsilon^{d-1}\int_0^a s^{q(d-1)}\int_\omega\bigg[&\big(1+(d-1)\varepsilon q R_\omega\big)|\partial_s u|^2\\
&+\bigg(\frac{1}{\varepsilon^2 s^{2q}} + \frac{qR_\omega}{s^2\varepsilon} + (d-1)\frac{q^2R_\omega^2}{s^2}\bigg)|\nabla_t u|^2\bigg]\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s\\
&-\varepsilon^{d-2}\int_0^a s^{q(d-2)}\int_{\partial\omega}\big|u(s,t)\big|^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s,
\end{aligned}$$

In order to deal with L^2-spaces without weights we additionally consider the unitary transform

$$\mathcal{V}_\varepsilon :\ L^2(\Pi_a) \to L^2\big(\Pi_a, \varepsilon^{d-1}s^{q(d-1)}\mathrm{d}s\,\mathrm{d}\mathcal{H}^{d-1}(t)\big),$$
$$\mathcal{V}_\varepsilon u(s,t) := \varepsilon^{-\frac{d-1}{2}}s^{-\frac{q(d-1)}{2}}u(s,t),$$

and the symmetric bilinear forms $\widetilde{p}^{\pm}_{\varepsilon,a}$ in $L^2(\Pi_a)$ defined on $\mathcal{V}_\varepsilon^{-1}\mathcal{D}_a \equiv \mathcal{D}_a$ by

$$\widetilde{p}^{\pm}_{\varepsilon,a}(u,u) := p^{\pm}_{\varepsilon,a}(\mathcal{V}_\varepsilon u, \mathcal{V}_\varepsilon u),$$

i.e. for $u \in \mathcal{D}_a$ one has

$$\begin{aligned}
\widetilde{p}^{-}_{\varepsilon,a}(u,u) = \int_0^a \int_\omega \Bigg[&\big(1-(d-1)\varepsilon q R_\omega\big)\Big(\partial_s u - \frac{q(d-1)}{2s}\,u\Big)^2 \\
&+ \left(\frac{1}{\varepsilon^2 s^{2q}} - \frac{qR_\omega}{s^2\varepsilon} - (d-1)\frac{q^2R_\omega^2}{s^2}\right)|\nabla_t u|^2\Bigg]\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&- \int_0^a \frac{1}{\varepsilon s^q}\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, s^{2q-2}} \int_{\partial\omega} |u(s,t)|\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s, \\
\widetilde{p}^{+}_{\varepsilon,a}(u,u) = \int_0^a \int_\omega \Bigg[&\big(1+(d-1)\varepsilon q R_\omega\big)\Big(\partial_s u - \frac{q(d-1)}{2s}\,u\Big)^2 \\
&+ \left(\frac{1}{\varepsilon^2 s^{2q}} + \frac{qR_\omega}{\varepsilon s^2} + (d-1)\frac{q^2R_\omega^2}{s^2}\right)|\nabla_t u|^2\Bigg]\mathrm{d}\mathcal{H}^{d-1}(t)\mathrm{d}s \\
&- \int_0^a \frac{1}{\varepsilon s^q}\int_{\partial\omega} u^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s.
\end{aligned}$$

Assuming that

$$\varepsilon \in (0,\varepsilon_0)$$

with some sufficiently small $\varepsilon_0 > 0$ we can find suitable constants $c_j > 0$, $j \in \{1,2\}$, with $c_j\varepsilon_0 < 1$ such that for all $s \in (0,a)$ one has

$$\frac{qR_\omega}{\varepsilon s^2} + (d-1)\frac{q^2R_\omega^2}{s^2} \le c_1\varepsilon \cdot \frac{1}{\varepsilon^2 s^{2q}},$$

which yields,

$$\begin{aligned}
\frac{1}{\varepsilon^2 s^{2q}} + \frac{qR_\omega}{\varepsilon s^2} + (d-1)\frac{q^2R_\omega^2}{s^2} &\le \frac{1+c_1\varepsilon}{\varepsilon^2 s^{2q}}, \\
\frac{1}{\varepsilon^2 s^{2q}} - \frac{qR_\omega}{\varepsilon s^2} - (d-1)\frac{q^2R_\omega^2}{s^2} &\ge \frac{1-c_1\varepsilon}{\varepsilon^2 s^{2q}},
\end{aligned}$$

and

$$\sqrt{1+\varepsilon^2 q^2\, R_\omega^2\, s^{2q-2}} \le \frac{1}{1-c_2\varepsilon}.$$

This gives, with $c_0 := (d-1)qR_\omega$,

$$\begin{aligned}
\widetilde{p}^{-}_{\varepsilon,a}(u,u) \ge{}& (1-c_0\varepsilon)\int_0^a\int_\omega \Big(\partial_s u - \frac{q(d-1)}{2s}\,u\Big)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&+ (1-c_1\varepsilon)\int_0^a \frac{1}{\varepsilon^2 s^{2q}}\int_\omega |\nabla_t u|^2\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&- \frac{1}{1-c_2\varepsilon}\int_0^a \frac{1}{\varepsilon s^q}\int_{\partial\omega} u^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s, \\
\widetilde{p}^{+}_{\varepsilon,a}(u,u) \le{}& (1+c_0\varepsilon)\int_0^a\int_\omega \Big(\partial_s u - \frac{q(d-1)}{2s}\,u\Big)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&+ (1+c_1\varepsilon)\int_0^a \frac{1}{\varepsilon^2 s^{2q}}\int_\omega |\nabla_t u|^2\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s - \int_0^a \frac{1}{\varepsilon s^q}\int_{\partial\omega} u^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s.
\end{aligned}$$

Due to $u \in \mathcal{D}_a$ the integration by parts in s gives

$$\int_0^a \frac{u\,\partial_s u}{s}\mathrm{d}s = \int_0^a \frac{u^2}{2s^2}\mathrm{d}s,$$

resulting in

$$\int_0^a \int_\omega \Big(\partial_s u - \frac{q(d-1)}{2s}\, u\Big)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s = \int_0^a \int_\omega \Big(|\partial_s u|^2 + \frac{H}{s^2}\, u^2\Big)\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s,$$

$$H := \frac{q^2(d-1)^2 - 2q(d-1)}{4} \equiv \frac{\big(q(d-1)-1\big)^2 - 1}{4}. \tag{12}$$

By taking

$$c := \max\{c_0, c_1, c_2\}$$

and by adjusting the value of ε_0 to have $c\varepsilon_0 < 1$ we arrive at the inequalities

$$h^-_{\varepsilon,a}(u,u) \le \widetilde{p}^-_{\varepsilon,a}(u,u), \qquad \widetilde{p}^+_{\varepsilon,a}(u,u) \le h^+_{\varepsilon,a}(u,u)$$

valid for all $\varepsilon \in (0,\varepsilon_0)$ and all $u \in \mathcal{D}_a$, where the symmetric bilinear forms $h^\pm_{\varepsilon,a}$ in $L^2(\Pi_a)$ are defined on $\mathcal{D}_a$ by

$$\begin{aligned}
h^-_{\varepsilon,a}(u,u) &= (1-c\varepsilon)\int_0^a \int_\omega \Big(|\partial_s u|^2 + \frac{H}{s^2}\, u^2\Big)\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&\quad + (1-c\varepsilon)\int_0^a \frac{1}{\varepsilon^2 s^{2q}} \int_\omega |\nabla_t u|^2\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&\quad - \frac{1}{1-c\varepsilon}\int_0^a \frac{1}{\varepsilon s^q}\int_{\partial\omega} u^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s, \\
h^+_{\varepsilon,a}(u,u) &= (1+c\varepsilon)\int_0^a \int_\omega \Big(|\partial_s u|^2 + \frac{H}{s^2}\, u^2\Big)\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&\quad + (1+c\varepsilon)\int_0^a \frac{1}{\varepsilon^2 s^{2q}} \int_\omega |\nabla_t u|^2\,\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s - \int_0^a \frac{1}{\varepsilon s^q}\int_{\partial\omega} u^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\,\mathrm{d}s.
\end{aligned}$$

Using the min-max principle one summarizes the above considerations as follows:

Lemma 9. *For any $a > 0$ there exist $\varepsilon_0 > 0$ and $c > 0$, with $c\varepsilon_0 < 1$, such that for any $j \in \mathbb{N}$ and any $\varepsilon \in (0,\varepsilon_0)$ there holds*

$$\mu^-_j(\varepsilon,a) \le \lambda_j(T_{\varepsilon,a}) \le \mu^+_j(\varepsilon,a) \quad \textit{with} \quad \mu^\pm_j(\varepsilon,a) := \inf_{\substack{S\subset\mathcal{D}_a \\ \dim S = j}} \sup_{u\in S\setminus\{0\}} \frac{h^\pm_{\varepsilon,a}(u,u)}{\|u\|^2_{L^2(\Pi_a)}}.$$

3.2. **Upper bound.** In this subsection we are going to obtain an upper bound for the quantities $\mu^+_j(\varepsilon,a)$ defined in Lemma 9. The analysis is based on the observation that for any $u \in \mathcal{D}_a$ one has

$$\begin{aligned}
h^+_{\varepsilon,a}(u,u) &= (1+c\varepsilon)\int_0^a \int_\omega \Big(|\partial_s u|^2 + \frac{H}{s^2}\, u^2\Big)\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&\quad + (1+c\varepsilon)\int_0^a \frac{1}{\varepsilon^2 s^{2q}}\bigg[\int_\omega |\nabla_t u|^2\,\mathrm{d}\mathcal{H}^{d-1}(t) - \frac{\varepsilon s^q}{1+c\varepsilon}\int_{\partial\omega} u^2\,\mathrm{d}\mathcal{H}^{d-2}(t)\bigg]\,\mathrm{d}s.
\end{aligned}$$

In other words, if we denote

$$\rho_\varepsilon(s) := \frac{s^q}{1+c\varepsilon},$$

then

$$\begin{aligned}
h^+_{\varepsilon,a}(u,u) &= (1+c\varepsilon)\int_0^a \int_\omega \Big(|\partial_s u|^2 + \frac{H}{s^2}\, u^2\Big)\mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\
&\quad + (1+c\varepsilon)\int_0^a \frac{1}{\varepsilon^2 s^{2q}} r^\omega_{\varepsilon\rho_\varepsilon(s)}\big(u(s,\cdot),u(s,\cdot)\big)\mathrm{d}s.
\end{aligned} \tag{13}$$

Let ψ_α be the positive and L^2-normalized eigenfunction of the Robin Laplacian R^ω_α for the first eigenvalue $\lambda_1(R^\omega_\alpha)$. In view of the above representation one is interested in $\psi_{\varepsilon\rho_\varepsilon(s)}$. By the assertion (iii) of Lemma 3, for any $\varepsilon > 0$ the map

$$(0, a) \ni s \mapsto \psi_{\varepsilon\rho_\varepsilon(s)} \in L^2(\omega)$$

is smooth. For any function $f \in W^{1,2}_{(0,a)}(0, a)$ consider the associated function

$$u : \ \Pi_a \ni (s, t) \mapsto f(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t). \tag{14}$$

Obviously u belongs to $L^2(\Pi_a)$, its weak derivatives in Π_a are given by

$$\begin{aligned}\partial_s u : (s, t) &\mapsto f'(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t) + f(s)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t),\\ \nabla_t u : (s, t) &\mapsto f(s)\nabla_t\psi_{\varepsilon\rho_\varepsilon(s)}(t),\end{aligned}$$

and, therefore, also belongs to $L^2(\Pi_a)$, which shows that $u \in \mathcal{D}_a$.

Lemma 10. *For any $a > 0$ and $\varepsilon_0 > 0$ there exists $B > 0$ such that for any $f \in W^{1,2}_{(0,a)}(0, a)$, any $\varepsilon \in (0, \varepsilon_0)$ and for u given by (14) it holds*

$$\begin{aligned}&\int_{\Pi_a} u^2 \mathrm{d}\mathcal{H}^d = \int_0^a f(s)^2 \mathrm{d}s,\\ &h^+_{\varepsilon,a}(u, u) \le (1 + c\varepsilon)\int_0^a \Big[f'(s)^2 + \Big(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1 + c\varepsilon)s^q} + B\Big)f(s)^2\Big]\mathrm{d}s.\end{aligned}$$

Proof. The normalization of $\psi_{\varepsilon\rho_\varepsilon(s)}$ gives

$$\begin{aligned}\int_\omega u(s, t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) &= f(s)^2 \int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = f(s)^2,\\ \int_{\Pi_a} u^2 \mathrm{d}\mathcal{H}^d &= \int_0^a \int_\omega u(s, t)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\mathrm{d}s = \int_0^a f(s)^2 \mathrm{d}s.\end{aligned}$$

Due to the choice of $\psi_{\varepsilon\rho_\varepsilon(s)}$ we have

$$r^\omega_{\varepsilon\rho_\varepsilon(s)}(u(s, \cdot), u(s, \cdot)) = \lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)}) \int_\omega u(s, t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = \lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)})f(s)^2.$$

Finally,

$$\begin{aligned}\int_\omega |\partial_s u|^2 \mathrm{d}\mathcal{H}^{d-1}(t) &= \int_\omega \Big(f'(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t) + f(s)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\Big)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\\ &= f'(s)^2 + 2f(s)f'(s)\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\mathrm{d}\mathcal{H}^{d-1}(t)\\ &\quad + f(s)^2\int_\omega \Big|\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\Big|^2 \mathrm{d}\mathcal{H}^{d-1}(t),\end{aligned}$$

and using

$$\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\,\mathrm{d}\mathcal{H}^{d-1}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}1 = 0$$

to eliminate the middle term on the right-hand side we obtain

$$\int_\omega |\partial_s u|^2 \mathrm{d}\mathcal{H}^{d-1}(t) = f'(s)^2 + W_\varepsilon f(s)^2, \qquad W_\varepsilon(s) := \int_\omega \Big|\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\Big|^2 \mathrm{d}\mathcal{H}^{d-1}(t).$$

The substitution of the previous identities into (13) yields

$$h^+_{\varepsilon,a}(u, u) = (1 + c\varepsilon)\int_0^a \left[f'(s)^2 + \left(W_\varepsilon(s) + \frac{H}{s^2} + \frac{\lambda_1\left(R^\omega_{\varepsilon\rho_\varepsilon(s)}\right)}{\varepsilon^2 s^{2q}}\right) f(s)^2\right] \mathrm{d}s. \tag{15}$$

Using Lemma 3(iv) we estimate, with a suitable $C_1 > 0$,

$$\frac{\lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)})}{\varepsilon^2 s^{2q}} \le \frac{-A_\omega \varepsilon\rho_\varepsilon(s) + C_1\varepsilon^2\rho_\varepsilon(s)^2}{\varepsilon^2 s^{2q}}$$
$$= -A_\omega \frac{\rho_\varepsilon(s)}{s^{2q}}\cdot\frac{1}{\varepsilon} + C_1\frac{\rho_\varepsilon(s)^2}{s^{2q}} \le -\frac{A_\omega}{\varepsilon(1+c\varepsilon)}\cdot\frac{1}{s^q} + C_1.$$

Furthermore,

$$W_\varepsilon(s) = \int_\omega \left(\frac{\mathrm{d}\psi_\sigma}{\mathrm{d}\sigma}(t)\Big|_{\sigma=\varepsilon\rho_\varepsilon(s)} \frac{\mathrm{d}\big(\varepsilon\rho_\varepsilon(s)\big)}{\mathrm{d}s}\right)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = \frac{q^2 s^{2q-2}}{(1+c\varepsilon)^2}\,\varepsilon^2 \left\|\frac{\mathrm{d}\psi_\sigma}{\mathrm{d}\sigma}\Big|_{\sigma=\varepsilon\rho_\varepsilon(s)}\right\|^2_{L^2(\omega)}.$$

By Lemma 3(iii) we can find $C_2 > 0$ such that

$$\left\|\frac{\mathrm{d}\psi_\sigma}{\mathrm{d}\sigma}\Big|_{\sigma=\varepsilon\rho_\varepsilon(s)}\right\|^2_{L^2(\omega)} \le C_2 \quad \text{for all } s\in(0,a) \text{ and } \varepsilon\in(0,\varepsilon_0),$$

therefore,

$$W_\varepsilon(s) \le \frac{C_2 q^2 s^{2q-2}}{(1+c\varepsilon)^2}\varepsilon^2 \le C_2 q^2 a^{2q-2}\varepsilon_0^2 =: C_3.$$

Making use of these inequalities in (15) one arrives at the claim with $B := C_1 + C_3$. □

Lemma 11. *For any $a > 0$ and $j \in \mathbb{N}$ there holds*

$$\lambda_j(T_{\varepsilon,a}) \le \left(\frac{A_\omega}{\varepsilon}\right)^{\frac{2}{2-q}} \lambda_j(L_1) + O\left(\frac{1}{\varepsilon}\right)^{\frac{q}{2-q}} \quad \textit{for } \varepsilon\to 0^+.$$

Proof. Let $j \in \mathbb{N}$ and $S \subset W^{1,2}_{(0,a)}(0,a)$ be a j-dimensional subspace. Due to the first identity in Lemma 10 the set

$$\widetilde{S} := \Big\{u:\ u(s,t) = f(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t) \text{ with } f\in S\Big\}$$

is a j-dimensional subspace of $\mathcal{D}_a$. Therefore,

$$\mu_j^+(\varepsilon,a) \le \sup_{u\in\widetilde{S}\setminus\{0\}} \frac{h^+_{\varepsilon,a}(u,u)}{\|u\|^2_{L^2(\Pi_a)}},$$

and using Lemma 10 one obtains

$$\frac{\mu_j^+(\varepsilon,a)}{1+c\varepsilon} \le \sup_{u\in S\setminus\{0\}} \frac{\displaystyle\int_0^a \left[f'(s)^2 + \left(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1+c\varepsilon)s^q}\right)f(s)^2\right]\mathrm{d}s}{\|f\|^2_{L^2(0,a)}} + B.$$

The constants c and B are independent of S, so one can take the infimum over all S as above to arrive at

$$\frac{\mu_j^+(\varepsilon,a)}{1+c\varepsilon} \le \inf_{\substack{S\subset W^{1,2}_{(0,a)}(0,a)\\ \dim S = j}} \sup_{u\in S\setminus\{0\}} \frac{\displaystyle\int_0^a \left[f'(s)^2 + \left(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1+c\varepsilon)s^q}\right)f(s)^2\right]\mathrm{d}s}{\|f\|^2_{L^2(0,a)}} + B.$$

The first summand on the right-hand side is exactly the characterization of the eigenvalue $\lambda_j\left(L_{\frac{A_\omega}{c(1+c\varepsilon)},a}\right)$ using the min-max principle, which yields

$$\mu_j^+(\varepsilon,a) \le (1+c\varepsilon)\lambda_j\left(L_{\frac{A_\omega}{\varepsilon(1+c\varepsilon)},a}\right) + (1+c\varepsilon)B.$$

With the help of the upper bound for $\lambda_j(T_{\varepsilon,a})$ and Lemma 4 with $\varepsilon \to 0^+$ we obtain, with some $K > 0$,

$$\lambda_j(T_{\varepsilon,a}) \le \mu_j^+(\varepsilon, a) \le (1 + c\varepsilon) \left[\left(\frac{A_\omega}{\varepsilon(1 + c\varepsilon)} \right)^{\frac{2}{2-q}} \lambda_j(L_1) + K \right] + (1 + c\varepsilon)B,$$

which gives the sought estimate. □

3.3. **Lower bound.** Similarly to the upper bound, the subsequent estimates for the numbers $\mu_j^-(\varepsilon, a)$ defined in Lemma 9 will be based on the observation that for any $u \in \mathcal{D}_a$ one has

$$\begin{aligned} h_{\varepsilon,a}^-(u, u) = (1 - c\varepsilon) \int_0^a \int_\omega \left(|\partial_s u|^2 + \frac{H}{s^2} u^2 \right) \mathrm{d}\mathcal{H}^{d-1}(t)\, \mathrm{d}s \\ + (1 - c\varepsilon) \int_0^a \frac{1}{\varepsilon^2 s^{2q}} r^\omega_{\varepsilon\rho_\varepsilon(s)} \big(u(s, \cdot), u(s, \cdot)\big) \mathrm{d}s, \qquad (16) \\ \text{with } \rho_\varepsilon(s) := \frac{s^q}{(1 - c\varepsilon)^2}. \end{aligned}$$

As above, let ψ_α be the positive and L^2-normalized eigenfunction of the Robin Laplacian R^ω_α for the first eigenvalue $\lambda_1(R^\omega_\alpha)$. We represent each function $u \in \mathcal{D}_a$ as

$$\begin{aligned} u &= v + w, \qquad (17) \\ v(s, t) &:= f(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t), \\ f(s) &:= \int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t) u(s, t)\, \mathrm{d}\mathcal{H}^{d-1}(t), \\ w &:= u - v. \end{aligned}$$

By construction one has $f \in W^{1,2}_{(0,a)}(0, a)$, and due to Lemma 3(iii) the weak derivatives of v in Π_a are given by

$$\begin{aligned} \partial_s v : (s, t) &\mapsto f'(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t) + f(s) \frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t), \qquad (18) \\ \nabla_t v : (s, t) &\mapsto f(s)\nabla_t \psi_{\varepsilon\rho_\varepsilon(s)}(t). \end{aligned}$$

In particular, the functions u, $\partial_s u$ and $\partial_{t_j} u$ belong to $L^2(\Pi_a)$, so one has $v, w \in \mathcal{D}_a$.

Lemma 12. *There exist a sufficiently small $\varepsilon_0 > 0$ and a sufficiently large $B > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $u \in \mathcal{D}_a$ decomposed as in (17) there holds*

$$\begin{aligned} &\|u\|^2_{L^2(\Pi_a)} = \|f\|^2_{L^2(0,a)} + \|w\|^2_{L^2(\Pi_a)}, \\ &\frac{h^-_{\varepsilon,a}(u, u)}{1 - c\varepsilon} \ge (1 - B\varepsilon) \int_0^a \left[f'(s)^2 + \left(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1 - B\varepsilon)^2 s^q} \right) f(s)^2 \right] \mathrm{d}s - B\|f\|^2_{L^2(0,a)}. \end{aligned}$$

Proof. Let us collect important properties of the decomposition (17). First,

$$\begin{aligned} \int_\omega v(s, t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = f(s)^2 \int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = f(s)^2, \\ \int_{\Pi_a} v^2 \mathrm{d}\mathcal{H}^d = \int_0^a \int_\omega v(s, t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) \mathrm{d}s = \int_0^a f(s)^2 \mathrm{d}s. \end{aligned}$$

Furthermore, by construction we have

$$\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t) w(s, t) \mathrm{d}\mathcal{H}^{d-1}(t) = 0, \qquad (19)$$

in particular,

$$\int_\omega v(s, t) w(s, t)\, \mathrm{d}\mathcal{H}^{d-1}(t) = f(s) \int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t) w(s, t)\, \mathrm{d}\mathcal{H}^{d-1}(t) = 0, \qquad (20)$$

therefore,

$$\begin{aligned}\int_\omega u(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) &= \int_\omega v(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) + 2\int_\omega v(s,t)w(s,t)\mathrm{d}\mathcal{H}^{d-1}(t) \\ &\quad + \int_\omega w(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) \\ &= \int_\omega v(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) + \int_\omega w(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) \\ &= f(s)^2 + \int_\omega w(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t), \\ \|u\|^2_{L^2(\Pi_a)} &= \|f\|^2_{L^2(0,a)} + \|w\|^2_{L^2(\Pi_a)}.\end{aligned}$$

Due to the choice of $\psi_{\varepsilon\rho_\varepsilon(s)}$ and the orthogonality (20), the spectral theorem for $R^\omega_{\varepsilon,\rho_\varepsilon(s)}$ gives

$$\begin{aligned}r^\omega_{\varepsilon\rho_\varepsilon(s)}\big(u(s,\cdot),u(s,\cdot)\big) &= r^\omega_{\varepsilon\rho_\varepsilon(s)}\big(v(s,\cdot),v(s,\cdot)\big) + r^\omega_{\varepsilon\rho_\varepsilon(s)}\big(w(s,\cdot),w(s,\cdot)\big) \\ &\geq \lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)})\big\|v(s,\cdot)\big\|^2_{L^2(\omega)} + \lambda_2(R^\omega_{\varepsilon\rho_\varepsilon(s)})\big\|w(s,\cdot)\big\|^2_{L^2(\omega)} \\ &= \lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)}) f(s)^2 + \lambda_2(R^\omega_{\varepsilon\rho_\varepsilon(s)})\big\|w(s,\cdot)\big\|^2_{L^2(\omega)}.\end{aligned}$$

Using these estimates in (16) one obtains

$$\begin{aligned}\frac{h^-_{\varepsilon,a}(u,u)}{1-c\varepsilon} &\geq \int_0^a \int_\omega |\partial_s u|^2 \mathrm{d}\mathcal{H}^{d-1}(t)\mathrm{d}s + \int_0^a \bigg(\frac{H}{s^2} + \frac{\lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)})}{\varepsilon^2 s^{2q}}\bigg) f(s)^2 \mathrm{d}s \\ &\quad + \int_0^a \int_\omega \bigg(\frac{H}{s^2} + \frac{\lambda_2(R^\omega_{\varepsilon\rho_\varepsilon(s)})}{\varepsilon^2 s^{2q}}\bigg) w(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\mathrm{d}s.\end{aligned}$$

We first use Lemma 3(iv) to estimate with a suitable $C_1 > 0$:

$$\begin{aligned}\frac{\lambda_1(R^\omega_{\varepsilon\rho_\varepsilon(s)})}{\varepsilon^2 s^{2q}} &\geq -\frac{A_\omega \varepsilon\rho_\varepsilon(s) + C_1\varepsilon^2\rho_\varepsilon(s)^2}{\varepsilon^2 s^{2q}} \\ &= -A_\omega \frac{\rho_\varepsilon(s)}{s^{2q}} \cdot \frac{1}{\varepsilon} - C_1 \frac{\rho_\varepsilon(s)^2}{s^{2q}} \geq -\frac{A_\omega}{\varepsilon(1-c\varepsilon)^2} \cdot \frac{1}{s^q} - \frac{C_1}{(1-c\varepsilon)^4}.\end{aligned}$$

In addition, using Lemma 3(v) we find some $C_2 > 0$ and, if necessary, decrease the value of ε_0 to obtain

$$\lambda_2(R^\omega_{\varepsilon\rho_\varepsilon(s)}) \geq C_2 \text{ for all } \varepsilon \in (0,\varepsilon_0) \text{ and } s \in (0,a).$$

Then for a suitable $C_3 > 0$ and an adjusted value of ε_0 one has

$$\frac{H}{s^2} + \frac{\lambda_2(R^\omega_{\varepsilon\rho_\varepsilon(s)})}{\varepsilon^2 s^{2q}} \geq \frac{C_3}{\varepsilon^2 s^{2q}} \text{ for all } \varepsilon \in (0,\varepsilon_0) \text{ and } s \in (0,a),$$

yielding

$$\begin{aligned}\frac{h^-_{\varepsilon,a}(u,u)}{1-c\varepsilon} &\geq \int_0^a \int_\omega |\partial_s u|^2 \mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s \\ &\quad + \int_0^a \bigg(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1-c\varepsilon)^2} \cdot \frac{1}{s^q} - \frac{C_1}{(1-c\varepsilon)^4}\bigg) f(s)^2 \mathrm{d}s \\ &\quad + \int_0^a \int_\omega \frac{C_3}{\varepsilon^2 s^{2q}} w(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s.\end{aligned} \tag{21}$$

Our next aim is to study the term containing $|\partial_s u|^2$ on the right-hand side of (21). By denoting

$$\begin{aligned}
I_1 &:= \int_\omega |\partial_s v|^2 \mathrm{d}\mathcal{H}^{d-1}(t),\\
I_2 &:= 2\int_\omega \partial_s v\cdot\partial_s w \,\mathrm{d}\mathcal{H}^{d-1}(t),\\
I_3 &:= \int_\omega |\partial_s w|^2 \mathrm{d}\mathcal{H}^{d-1}(t),
\end{aligned}$$

we arrive at the decomposition

$$\int_\omega |\partial_s u|^2 \mathrm{d}\mathcal{H}^{d-1}(t) = I_1 + I_2 + I_3. \tag{22}$$

The term I_1 is analyzed in a straightforward way:

$$\begin{aligned}
I_1 &= \int_\omega \Big(f'(s)\psi_{\varepsilon\rho_\varepsilon(s)}(t) + f(s)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\Big)^2 \mathrm{d}\mathcal{H}^{d-1}(t)\\
&= f'(s)^2 + 2f(s)f'(s)\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\mathrm{d}\mathcal{H}^{d-1}(t)\\
&\quad + f(s)^2\int_\omega \Big|\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\Big|^2 \mathrm{d}\mathcal{H}^{d-1}(t),
\end{aligned}$$

and using

$$\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\,\mathrm{d}\mathcal{H}^{d-1}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)^2 \mathrm{d}\mathcal{H}^{d-1}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}1 = 0$$

to eliminate the middle term on the right-hand side, and by noting that last term is nonnegative we obtain

$$I_1 \geq f'(s)^2. \tag{23}$$

As a consequence of (18), we have

$$\begin{aligned}
I_2 &= I_2' + I_2'',\\
I_2' &= 2f'(s)\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)\partial_s w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t),\\
I_2'' &= 2f(s)\int_\omega \frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\partial_s w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t).
\end{aligned}$$

Using the orthogonality (19) we obtain

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}s}\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t)\\
&= \int_\omega \frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t) + \int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}\partial_s w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t),
\end{aligned}$$

therefore,

$$\int_\omega \psi_{\varepsilon\rho_\varepsilon(s)}(t)\partial_s w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t) = -\int_\omega \frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)w(s,t)\,\mathrm{d}\mathcal{H}^{d-1}(t).$$

This allows one to estimate I_2' by

$$\begin{aligned}
|I_2'| &= \Big|\int_\omega 2f'(s)\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)w(s,t)\mathrm{d}\mathcal{H}^{d-1}(t)\Big|\\
&\leq W_\varepsilon(s)f'(s)^2 + \int_\omega w(s,t)^2\mathrm{d}\mathcal{H}^{d-1}(t)
\end{aligned}$$

$$\text{with } W_\varepsilon(s) := \int_\omega \Big|\frac{\mathrm{d}\psi_{\varepsilon\rho_\varepsilon(s)}}{\mathrm{d}s}(t)\Big|^2 \mathrm{d}\mathcal{H}^{d-1}(t).$$

Furthermore,

$$|I_2''| \le \frac{W_\varepsilon(s)}{\varepsilon} f(s)^2 + \varepsilon \int_\omega \partial_s w(s,t)^2 \mathrm{d}\mathcal{H}^{d-1}(t).$$

We have

$$\begin{aligned} W_\varepsilon(s) &= \int_\omega \left(\frac{\mathrm{d}\psi_\sigma}{\mathrm{d}\sigma}(t)\Big|_{\sigma=\varepsilon\rho_\varepsilon(s)} \frac{\mathrm{d}\big(\varepsilon\rho_\varepsilon(s)\big)}{\mathrm{d}s}\right)^2 \mathrm{d}\mathcal{H}^{d-1}(t) \\ &= \frac{q^2 s^{2q-2}}{(1-c\varepsilon)^4}\,\varepsilon^2 \left\| \frac{\mathrm{d}\psi_\sigma}{\mathrm{d}\sigma}\Big|_{\sigma=\varepsilon\rho_\varepsilon(s)} \right\|^2_{L^2(\omega)}, \end{aligned}$$

and by Lemma 3(iii) we can find $C_4 > 0$ such that

$$\left\| \frac{\mathrm{d}\psi_\sigma}{\mathrm{d}\sigma}\Big|_{\sigma=\varepsilon\rho_\varepsilon(s)} \right\|^2_{L^2(\omega)} \le C_4 \quad \text{for all } s \in (0,a) \text{ and } \varepsilon \in (0,\varepsilon_0),$$

then for some $C_5 > 0$ it holds

$$W_\varepsilon(s) \le C_5 \varepsilon^2 \text{ for all } s \in (0,a) \text{ and } \varepsilon \in (0,\varepsilon_0).$$

Then

$$\begin{aligned} |I_2'| &\le C_5\varepsilon^2 f'(s)^2 + \|w(s,\cdot)\|^2_{L^2(\omega)}, \\ |I_2''| &\le C_5\varepsilon f(s)^2 + \varepsilon\|\partial_s w(s,\cdot)\|^2_{L^2(\omega)}, \end{aligned}$$

and

$$I_2 \ge -|I_2'| - |I_2''| \ge C_5\varepsilon^2 f'(s)^2 - C_5\varepsilon f(s)^2 - \varepsilon\|\partial_s w(s,\cdot)\|^2_{L^2(\omega)} - \|w(s,\cdot)\|^2_{L^2(\omega)}.$$

By using the last inequality and (23) in (22) we arrive at

$$\begin{aligned} \|\partial_s u(s,\cdot)\|^2_{L^2(\omega)} \ge (1-C_5\varepsilon^2)f'(s)^2 - C_5\varepsilon f(s)^2 \\ + (1-\varepsilon)\|\partial_s w(s,\cdot)\|^2_{L^2(\omega)} - \|w(s,\cdot)\|^2_{L^2(\omega)}. \end{aligned}$$

By using the last inequality to estimate the first summand on the right-hand side of (21) we obtain

$$\begin{aligned} \frac{h^-_{\varepsilon,a}(u,u)}{1-c\varepsilon} \ge \int_0^a \left[(1-C_5\varepsilon^2)f'(s)^2 + \left(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1-c\varepsilon)^2 s^q} - \frac{C_1}{(1-c\varepsilon)^4} - C_5\varepsilon \right) f(s)^2 \right] \mathrm{d}s \\ + \int_0^a \int_\omega \left[(1-\varepsilon)\partial_s w(s,t)^2 + \left(\frac{C_3}{\varepsilon^2 s^{2q}} - 1 \right) w(s,t)^2 \right] \mathrm{d}\mathcal{H}^{d-1}(t)\,\mathrm{d}s. \end{aligned}$$

We additionally adjust ε_0 such that the expression in the last integral becomes non-negative for all $s \in (0,a)$ and all $\varepsilon \in (0,\varepsilon_0)$ and choose $C_6 > 0$ to have

$$\frac{C_1}{(1-c\varepsilon)^4} + C_5\varepsilon \le C_6 \text{ for all } \varepsilon \in (0,\varepsilon_0),$$

which results in

$$\frac{h^-_{\varepsilon,a}(u,u)}{1-c\varepsilon} \ge \int_0^a \left[(1-C_5\varepsilon^2)f'(s)^2 + \left(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon(1-c\varepsilon)^2 s^q} - C_6 \right) f(s)^2 \right] \mathrm{d}s. \tag{24}$$

Recall that due to the explicit expression (12) for H and the one-dimensional Hardy inequality we have

$$\begin{aligned} \frac{\big(q(d-1)-1\big)^2}{4} \int_0^a \frac{1}{s^2} f(s)^2 \mathrm{d}s &\le \int_0^a \left(f'(s)^2 + \frac{\big(q(d-1)-1\big)^2 - 1}{4s^2} f(s)^2 \right) \mathrm{d}s \\ &\equiv \int_0^a \left(f'(s)^2 + \frac{H}{s^2} f(s)^2 \right) \mathrm{d}s, \end{aligned}$$

yielding

$$\int_0^a \frac{1}{s^2} f(s)^2 \mathrm{d}s \le C_7 \int_0^a \Big(f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big) \mathrm{d}s, \quad C_7 := \frac{4}{\big(q(d-1)-1\big)^2}.$$

This implies

$$\begin{aligned}
&\int_0^a \Big[(1 - C_5\varepsilon^2) f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big] \mathrm{d}s \\
&\quad = (1 - C_5\varepsilon^2) \int_0^a \Big(f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big) \mathrm{d}s + H C_5 \varepsilon^2 \int_0^a \frac{1}{s^2} f(s)^2 \mathrm{d}s \\
&\quad \ge (1 - C_5\varepsilon^2) \int_0^a \Big(f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big) \mathrm{d}s - |H| C_5 C_7 \varepsilon^2 \int_0^a \Big(f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big) \mathrm{d}s \\
&\quad = (1 - C_8\varepsilon^2) \int_0^a \Big(f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big) \mathrm{d}s \quad \text{with } C_8 := C_5 + |H| C_5 C_7.
\end{aligned}$$

By using this inequality on the right-hand side of (24) we arrive at

$$\frac{h^-_{\varepsilon,a}(u,u)}{1 - c\varepsilon} \ge (1 - C_8\varepsilon^2) \int_0^a \Big(f'(s)^2 + \frac{H}{s^2} f(s)^2 \Big) \mathrm{d}s - \int_0^a \frac{A_\omega}{\varepsilon s^q (1 - c\varepsilon)^2} f(s)^2 \mathrm{d}s - C_6 \|f\|^2_{L^2(0,a)}.$$

We assume additionally $\varepsilon_0 \in (0,1)$, then $\varepsilon^2 < \varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$, and we obtain the claim by choosing any $B \ge \max\{C_6, C_8\}$ such that

$$\frac{1}{(1 - c\varepsilon)^2} \le \frac{1}{1 - B\varepsilon} \text{ for all } \varepsilon \in (0, \varepsilon_0) \qquad \square$$

holds.

Lemma 13. *For any $a > 0$ and any $j \in \mathbb{N}$ one has*

$$\lambda_j(T_{\varepsilon,a}) \le \Big(\frac{A_\omega}{\varepsilon} \Big)^{\frac{2}{2-q}} \lambda_j(L_1) + O\Big(\frac{1}{\varepsilon} \Big)^{\frac{q}{2-q}} \quad \textit{for } \varepsilon \to 0^+.$$

Proof. Due to the first identity in Lemma 12, the map $u \mapsto (f, w)$ defined by (17) is uniquely extended to an isometry

$$J: \ L^2(\Pi_a) \to L^2(0,a) \oplus L^2(\Pi_a).$$

Consider the symmetric bilinear form m_ε in $L^2(0,a) \oplus L^2(\Pi_a)$ given by

$$m_\varepsilon\big((f,w),(f,w)\big) := (1 - B\varepsilon) \int_0^a \Big[f'(s)^2 + \Big(\frac{H}{s^2} - \frac{A_\omega}{\varepsilon (1 - B\varepsilon)^2 s^q} \Big) f(s)^2 \Big] \mathrm{d}s - B \|f\|^2_{L^2(0,a)}$$

on $\mathcal{D}(m_\varepsilon) := W^{1,2}_{(0,a)}(0,a) \oplus L^2(\Pi_a)$, then the inequality in Lemma 12 can be read as

$$\frac{h^-_{\varepsilon,a}(u,u)}{1 - c\varepsilon} \ge m_\varepsilon(Ju, Ju) \text{ for all } u \in \mathcal{D}_a.$$

Further remark that the closure of m_ε is the bilinear form corresponding to the self-adjoint operator

$$M_\varepsilon := \Big((1 - B\varepsilon) L_{\frac{A_\omega}{\varepsilon(1-B\varepsilon)^2}, a} - B \Big) \oplus 0.$$

Let $j \in \mathbb{N}$. For each j-dimensional subspace $S \subset \mathcal{D}_a$ the set $J(S)$ is a j-dimensional subspace in the domain of m_ε, and due to Lemma 9 we have

$$
\begin{aligned}
\frac{\lambda_j(T_{\varepsilon,a})}{1-c\varepsilon} \geq \frac{\mu_j^-(\varepsilon,a)}{1-c\varepsilon} &\equiv \inf_{\substack{S\subset\mathcal{D}_a\\ \dim S=j}} \sup_{u\in S\setminus\{0\}} \frac{h_{\varepsilon,a}^-(u,u)}{\|u\|^2_{L^2(\Pi_a)}} \\
&\geq \inf_{\substack{S\subset\mathcal{D}_a\\ \dim S=j}} \sup_{u\in S\setminus\{0\}} \frac{m_\varepsilon(Ju,Ju)}{\|Ju\|^2_{L^2(0,a)\oplus L^2(\Pi_a)}} \\
&\equiv \inf_{\substack{S\subset\mathcal{D}_a\\ \dim S=j}} \sup_{z\in J(S)\setminus\{0\}} \frac{m_\varepsilon(z,z)}{\|z\|^2_{L^2(0,a)\oplus L^2(\Pi_a)}} \\
&\geq \inf_{\substack{S'\subset\mathcal{D}(m_\varepsilon)\\ \dim S'=j}} \sup_{z\in S'\setminus\{0\}} \frac{m_\varepsilon(z,z)}{\|z\|^2_{L^2(0,a)\oplus L^2(\Pi_a)}} = \lambda_j(M_\varepsilon).
\end{aligned} \tag{25}
$$

For $\varepsilon \to 0^+$ we have, due to Lemma 4,

$$
\begin{aligned}
\lambda_j\Big((1-B\varepsilon)L_{\frac{A_\omega}{\varepsilon(1-B\varepsilon)^2},a} - B\Big) &= (1-B\varepsilon)\lambda_j(L_{\frac{A_\omega}{\varepsilon(1-B\varepsilon)^2},a}) - B \\
&\geq (1-B\varepsilon)\Big(\frac{A_\omega}{\varepsilon(1-B\varepsilon)^2}\Big)^{\frac{2}{2-q}}\lambda_j(L_1) - B \\
&= \left(\frac{A_\omega}{\varepsilon}\right)^{\frac{2}{2-q}} \lambda_j(L_1) + O\left(\frac{1}{\varepsilon}\right)^{\frac{q}{2-q}},
\end{aligned}
$$

hence,

$$
\begin{aligned}
\lambda_j(M_\varepsilon) = \lambda_j\left(\Big[(1-B\varepsilon)L_{\frac{A_\omega}{\varepsilon(1-B\varepsilon)^2},a} - B\Big] \oplus 0\right) &\geq \min\left\{\lambda_j\Big((1-B\varepsilon)L_{\frac{A_\omega}{\varepsilon(1-B\varepsilon)^2},a} - B\Big), 0\right\} \\
\geq \min\left\{\left(\frac{A_\omega}{\varepsilon}\right)^{\frac{2}{2-q}} \lambda_j(L_1) + O\left(\frac{1}{\varepsilon}\right)^{\frac{q}{2-q}}, 0\right\} &= \left(\frac{A_\omega}{\varepsilon}\right)^{\frac{2}{2-q}} \lambda_j(L_1) + O\left(\frac{1}{\varepsilon}\right)^{\frac{q}{2-q}},
\end{aligned}
$$

and the substitution into (25) completes the proof. □

By combining the upper bound of Lemma 11 and the lower bound of Lemma 13 and by recalling the convention (11) we arrive at the main result of this section:

Corollary 14. *For any $a > 0$ and $j \in \mathbb{N}$ there holds*

$$
\lambda_j(T^{1,D}_{\varepsilon,(0,a)}) = \left(\frac{A_\omega}{\varepsilon}\right)^{\frac{2}{2-q}} \lambda_j(L_1) + O\left(\frac{1}{\varepsilon}\right)^{\frac{q}{2-q}} \quad \textit{for } \varepsilon \to 0^+.
$$

4. Truncations and the proof of the main theorem

In this section we prove Theorem 2. The argument will be based on a series of truncations combined with the spectral analysis of the operators $T^{\alpha,N/D}_{\varepsilon,I}$ from the previous sections.

Choose $\delta > 0$ for Ω as in Definition 1, then pick any $a \in (0,\delta)$ and consider the sets

$$
\Omega_a := \Omega \cap \Big[(-a,a)\times(-\delta,\delta)^{d-1}\Big], \qquad \Omega'_a := \Omega \setminus \overline{\Omega_a}.
$$

By assumption on Ω there holds $\Omega_a = V_{1,(0,a)}$, while Ω'_a is a bounded Lipschitz domain.

Lemma 15. *For any $j \in \mathbb{N}$ one has*

$$
\lambda_j(R^\Omega_\alpha) \leq A_\omega^{\frac{2}{2-q}} \lambda_j(L_1) \alpha^{\frac{2}{2-q}} + O\Big(\alpha^{\frac{2}{2-q}-(q-1)}\Big) \quad \textit{for } \alpha \to +\infty.
$$

Proof. As each function from $\widehat{W}_0^{1,2}(V_{(1,(0,a)})$ can be extended by zero to a function in $W^{1,2}(\Omega)$, the min-max principle and the scaling (6) imply that for any $j\in\mathbb{N}$ and any $\alpha>0$ there holds

$$\lambda_j(R_\alpha^\Omega)\le\lambda_j(T^{\alpha,D}_{1,(0,a)})=\alpha^2\lambda_j(T^{1,D}_{\alpha^{1-q},(0,\alpha a)}).$$

Now assume that $\alpha>1$ and remark that each function from $\widehat{W}_0^{1,2}(V_{1,(0,a)})$ can be extended by zero to a function in $\widehat{W}_0^{1,2}(V_{1,(0,\alpha a)})$, so the min-max principle implies

$$\lambda_j(T^{1,D}_{\alpha^{1-q},(0,\alpha a)})\le\lambda_j(T^{1,D}_{\alpha^{1-q},(0,a)}).$$

Using Corollary 14 with $\varepsilon:=\alpha^{1-q}$ and $\alpha\to+\infty$ we obtain

$$\lambda_j(T^{1,D}_{\alpha^{1-q},(0,a)})=\left(\frac{A_\omega}{\alpha^{1-q}}\right)^{\frac{2}{2-q}}\lambda_j(L_1)+O\left(\frac{1}{\alpha^{1-q}}\right)^{\frac{q}{2-q}}=A_\omega^{\frac{2}{2-q}}\alpha^{\frac{2q-2}{2-q}}\lambda_j(L_1)+O\left(\alpha^{\frac{q^2-q}{2-q}}\right),$$

and the preceding inequalities yield

$$\begin{aligned}\lambda_j(R_\alpha^\Omega)\le\alpha^2\lambda_j(T^{1,D}_{\alpha^{1-q},(0,a)})&=\alpha^2\Big[A_\omega^{\frac{2}{2-q}}\alpha^{\frac{2q-2}{2-q}}\lambda_j(L_1)+O\Big(\alpha^{\frac{q^2-q}{2-q}}\Big)\Big]\\&=A_\omega^{\frac{2}{2-q}}\lambda_j(L_1)\alpha^{\frac{2}{2-q}}+O\Big(\alpha^{\frac{2}{2-q}-(q-1)}\Big).\end{aligned}$$

□

Obtaining a lower bound requires slightly more work.

Lemma 16. *For any $j\in\mathbb{N}$ one has*

$$\lambda_j(R_\alpha^\Omega)\ge A_\omega^{\frac{2}{2-q}}\lambda_j(L_1)\alpha^{\frac{2}{2-q}}+O\Big(\alpha^{\frac{2}{2-q}-(q-1)}\Big)\ \text{for}\ \alpha\to+\infty.$$

Proof. Let $j\in\mathbb{N}$ be fixed. For any $u\in W^{1,2}(\Omega)$ and any $\alpha>0$ we have

$$r_\alpha^\Omega(u,u)\ge t^{\alpha,N}_{1,(0,a)}(u|_{\Omega_a},u|_{\Omega_a})+r_\alpha^{\Omega'_a}(u|_{\Omega'_a},u|_{\Omega'_a}),$$

and the min-max principle shows that for any $\alpha>0$ it holds

$$\lambda_j(R_\alpha^\Omega)\ge\lambda_j(T^{\alpha,N}_{1,(0,a)}\oplus R_\alpha^{\Omega'_a}).$$

Remark that

$$\lambda_j\Big(T^{\alpha,N}_{1,(0,a)}\oplus R_\alpha^{\Omega'_a}\Big)\ge\min\Big\{\lambda_j(T^{\alpha,N}_{1,(0,a)}),\lambda_1(R_\alpha^{\Omega'_a})\Big\}.$$

As Ω'_a is a bounded Lipschitz domain, there is $c>0$ such that $\lambda_1(R_\alpha^{\Omega'_a})\ge-c\alpha^2$ for all sufficiently large α (as discussed in the introduction). Due to the scaling (6) we have

$$\lambda_j(T^{\alpha,N}_{1,(0,a)})=\alpha^2\lambda_j(T^{1,N}_{\alpha^{1-q},(0,a\alpha)}),$$

and by putting all together we arrive at

$$\lambda_j(R_\alpha^\Omega)\ge\min\Big\{\lambda_j(T^{1,N}_{\alpha^{1-q},(0,a\alpha)}),-c\Big\}\alpha^2\ \text{for all sufficiently large}\ \alpha. \tag{26}$$

From now on we assume that $\alpha>1$. To study the eigenvalues of $T^{1,N}_{\alpha^{1-q},(0,a\alpha)}$ we pick smooth functions $\chi_1,\chi_2\in C^\infty(0,\infty)$ such that

$$\chi_1(s)=0\ \text{for all}\ s>\frac{2a}{3},\qquad\chi_2(s)=0\ \text{for all}\ s<\frac{a}{3},\qquad\chi_1^2+\chi_2^2=1.$$

Due to Lemma 5 the set $W^{1,2}_{(0,\infty)}(V_{\alpha^{1-q},(0,a\alpha)})$ is a core domain of $t^{1,N}_{\alpha^{1-q},(0,a\alpha)}$, and for any $u \in W^{1,2}_{(0,\infty)}(V_{\alpha^{1-q},(0,a\alpha)})$ one has

$$\begin{aligned}
(x_1,x') &\mapsto \chi_1(x_1)u(x_1,x') \in W^{1,2}_{(0,a)}(V_{\alpha^{1-q},(0,a)}),\\
(x_1,x') &\mapsto \chi_2(x_1)u(x_1,x') \in W^{1,2}(V_{\alpha^{1-q},(\frac{a}{3},a\alpha)}),\\
\|\chi_1 u\|^2_{L^2(V_{\alpha^{1-q},(0,a)})} + \|\chi_2 u\|^2_{L^2(V_{\alpha^{1-q},(\frac{a}{3},a\alpha)})} &= \|u\|^2_{L^2(V_{\alpha^{1-q},(0,a\alpha)})},\\
t^{1,N}_{\alpha^{1-q},(0,a\alpha)}(u,u) &= t^{1,D}_{\alpha^{1-q},(0,a)}(\chi_1 u,\chi_1 u) + t^{1,N}_{\alpha^{1-q},(\frac{a}{3},a\alpha)}(\chi_2 u,\chi_2 u)\\
&\quad - \int_{V_{\alpha^{1-q},(0,a\alpha)}} (|\nabla\chi_1|^2 + |\nabla\chi_2|^2)u^2 \mathrm{d}\mathcal{H}^d,
\end{aligned}$$

and by denoting $B := \big\||\nabla\chi_1|^2 + |\nabla\chi_2|^2\big\|_\infty$ and using the min-max-principle we arrive at

$$\begin{aligned}
\lambda_j(T^{1,N}_{\alpha^{1-q},(0,a\alpha)}) &\geq \lambda_j\Big(T^{1,D}_{\alpha^{1-q},(0,a)} \oplus T^{1,N}_{\alpha^{1-q},(\frac{a}{3},a\alpha)}\Big) - B\\
&\geq \min\Big\{\lambda_j\Big(T^{1,D}_{\alpha^{1-q},(0,a)}\Big), \lambda_1\Big(T^{1,N}_{\alpha^{1-q},(\frac{a}{3},a\alpha)}\Big)\Big\} - B.
\end{aligned}$$

For some fixed $C > 0$ and all sufficiently large α we have, due to Lemma 8 and Corollary 14,

$$\lambda_j\Big(T^{1,D}_{\alpha^{1-q},(0,a)}\Big) = A_\omega^{\frac{2}{2-q}} \alpha^{\frac{2q-2}{2-q}} \lambda_j(L_1) + O\Big(\alpha^{\frac{q^2-q}{2-q}}\Big) < -C\alpha^{q-1} \leq \lambda_1\Big(T^{1,N}_{\alpha^{1-q},(\frac{a}{3},a\alpha)}\Big),$$

because

$$\frac{2q-2}{2-q} = \frac{2}{2-q}(q-1) > q-1.$$

This yields for all large α

$$\lambda_j(T^{1,N}_{\alpha^{1-q},(0,a\alpha)}) \geq \lambda_j\Big(T^{1,D}_{\alpha^{1-q},(0,a)}\Big) - B = A_\omega^{\frac{2}{2-q}} \alpha^{\frac{2q-2}{2-q}} \lambda_j(L_1) + O\Big(\alpha^{\frac{q^2-q}{2-q}}\Big),$$

and using (26) we obtain

$$\begin{aligned}
\lambda_j(R^\Omega_\alpha) &\geq \min\Big\{\lambda_j(T^{1,N}_{\alpha^{1-q},(0,a\alpha)}), -c\Big\}\alpha^2 \geq \min\Big\{A_\omega^{\frac{2}{2-q}} \alpha^{\frac{2q-2}{2-q}} \lambda_j(L_1) + O\Big(\alpha^{\frac{q^2-q}{2-q}}\Big), -c\Big\}\alpha^2\\
&= \Big[A_\omega^{\frac{2}{2-q}} \alpha^{\frac{2q-2}{2-q}} \lambda_j(L_1) + O\Big(\alpha^{\frac{q^2-q}{2-q}}\Big)\Big]\alpha^2 = A_\omega^{\frac{2}{2-q}} \lambda_j(L_1)\alpha^{\frac{2}{2-q}} + O\Big(\alpha^{\frac{2}{2-q}-(q-1)}\Big). \qquad \square
\end{aligned}$$

The claim of Theorem 2 follows by combining the upper bound of Lemma 15 with the lower bound of Lemma 16.

Acknowledgments

F. Sk is supported by the Alexander von Humboldt Foundation, Germany.

References

[1] G. Acosta, M. G. Armentano, R. G. Durán, A. L. Lombardi: *Nonhomogeneous Neumann problem for the Poisson equation in domains with an external cusp.* J. Math. Anal. Appl. **310** (2005) 397–411.

[2] P. R. S. Antunes, P. Freitas, D. Krejčiřík: *Bounds and extremal domains for Robin eigenvalues with negative boundary parameter.* Adv. Calc. Var. **10** (2017) 357–380.

[3] V. Bruneau, N. Popoff: *On the negative spectrum of the Robin Laplacian in corner domains.* Anal. PDE **9** (2016) 1259–1283.

[4] D. Bucur, P. Freitas, J. B. Kennedy: *The Robin problem.* A. Henrot (ed.): *Shape optimization and spectral theory.* De Gruyter Open, Warsaw, 2017, pp. 78–119.

[5] D. Daners: *Principal eigenvalues for generalised indefinite Robin problems.* Potential Anal. **38** (2013) 1047–1069.

[6] D. Daners, J. B. Kennedy: *On the asymptotic behaviour of the eigenvalues of a Robin problem.* Differ. Integr. Equ. **23** (2010) 659–669.

[7] B. Helffer, A. Kachmar: *Eigenvalues for the Robin Laplacian in domains with variable curvature.* Tran. Amer. Math. Soc. **369** (2017) 3253–3287.

[8] A. Kachmar, P. Keraval, N. Raymond: *Weyl formulae for the Robin Laplacian in the semiclassical limit.* Confl. Math. **8**:2 (2016) 39–57.

[9] M. Khalile, T. Ourmières-Bonafos, K. Pankrashkin: *Effective operators for Robin eigenvalues in domains with corners.* Ann. Inst. Fourier **70** (2020) 2215–2301.

[10] H. Kovařík, K. Pankrashkin: *Robin eigenvalues on domains with peaks.* J. Differential Equations **267** (2019) 1600–1630.

[11] M. Levitin, L. Parnovski: *On the principal eigenvalue of a Robin problem with a large parameter.* Math. Nachr. **281** (2008) 272–281.

[12] V. G. Maz'ya, S. V. Poborchi: *Differentiable functions on bad domains.* World Scientific, Singapore et al., 1997.

[13] S. A. Nazarov, J. Taskinen: *Spectral anomalies of the Robin Laplacian in non-Lipschitz domains.* J. Math. Sci. Univ. Tokyo **20** (2013) 27–90.

[14] K. Pankrashkin, N. Popoff: *Mean curvature bounds and eigenvalues of Robin Laplacians.* Calc. Var. PDE **54** (2015) 1947–1961.

[15] K. Pankrashkin, M. Vogel: *Asymptotics of Robin eigenvalues on sharp infinite cones.* J. Spectr. Theory **13**:1 (2023) 201–241.

[16] J. Rosenberg: *Applications of analysis on Lipschitz manifolds.* M. Cowling, C. Meaney, W. Moran (eds.): *Miniconferences on harmonic analysis and operator algebras (Canberra, 1987).* Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 16, Austral. Nat. Univ., Canberra, 1988, pp. 269–283.

[17] M. Vogel: *Asymptotics of Robin eigenvalues for non-isotropic peaks.* J. Math. Anal. Appl. **532**:2 (2024) 127953.

K. Pankrashkin, Carl von Ossietzky Universität Oldenburg, Fakultät V, Institut für Mathematik, Ammerländer Heerstrasse 114–118, 26129 Oldenburg, Germany

Email address: konstantin.pankrashkin@uol.de

F. Sk, Carl von Ossietzky Universität Oldenburg, Fakultät V, Institut für Mathematik, Ammerländer Heerstrasse 114–118, 26129 Oldenburg, Germany

Email address: firoj.sk@uol.de

M. Vogel, Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, 44227 Dortmund, Germany

Email address: marco.vogel@math.tu-dortmund.de