

# Tail-dependence properties of some new types of copula models (part II)

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**Abstract** We continue the investigation of the tail-dependence behaviour of some new types of copula models, published recently in [5] and [6].

**1. Introduction.** For the sake of simplicity, we concentrate our investigations to the two-dimensional case. Let  $U_1, U_2$  be standard random variables, i.e. they follow a uniform distribution over the interval  $[0,1]$  each. Let further  $T_1, T_2$  be real continuous functions over  $\mathbb{R}^2$  and  $W_1 = T_1(U_1, U_2)$ ,  $W_2 = T_2(U_1, U_2)$ . If  $W_1, W_2$  already follow a continuous uniform distribution over  $[0,1]$  each, then  $(W_1, W_2)$  is a representative of a two-dimensional copula. Otherwise,  $(V_1, V_2) := (F_1(W_1), F_2(W_2))$  is a representative of a two-dimensional copula if  $F_i$  denotes the continuous c.d.f. of  $W_i$ ,  $i = 1, 2$ .

Of particular interest especially for financial markets or risk management is the tail dependence of copulas which was explicitly treated for dependence-of-unity copulas in [2], [3] and [4], and for the new approach in [6], which we shall continue here. The simplest definition of the coefficient  $\lambda_U$  of upper and  $\lambda_L$  of lower tail dependence is

$$\lambda_U = \lim_{t \uparrow 1} \frac{P(W_1 > F_1^{-1}(t), W_2 > F_2^{-1}(t))}{1-t}, \quad \lambda_L = \lim_{t \downarrow 0} \frac{P(W_1 \leq F_1^{-1}(t), W_2 \leq F_2^{-1}(t))}{t}, \quad \text{see e.g. [1], Def. 7.36, p.247.}$$

In case that  $F_1 = F_2$ , i.e.  $W_1$  and  $W_2$  have same distribution, we also have

$$\lambda_U = \lim_{s \uparrow \infty} \frac{P(W_1 > s, W_2 > s)}{1-F(s)}, \quad \lambda_L = \lim_{s \downarrow -\infty} \frac{P(W_1 \leq s, W_2 \leq s)}{F(s)}.$$

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## 2. Particular Cases.

**Case 1.** Here we consider the choice

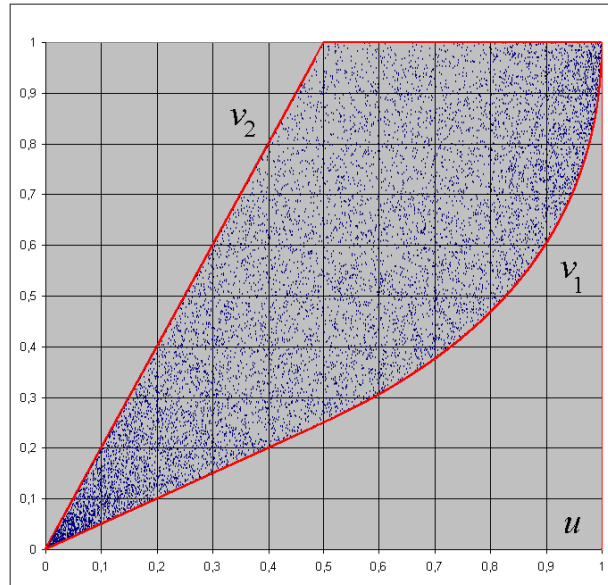
$$W_1 = T_1(U_1, U_2) = U_1 + U_2, \quad W_2 = T_2(U_1, U_2) = \max(U_1, U_2).$$

It is easy to see that the corresponding c.d.f.'s are given by

$$F_1(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x \leq 1 \\ 1 - 2\left(1 - \frac{x}{2}\right)^2, & 1 \leq x \leq 2. \end{cases} \quad \text{and}$$

$$F_2(x) = x^2, \quad 0 \leq x \leq 1.$$

The following graph shows 10.000 simulations of  $(V_1, V_2) = (F_1(W_1), F_2(W_2))$ .



The red lines  $(u, v)$  represent the (sharp) lower and upper envelopes of the copula, which are given by

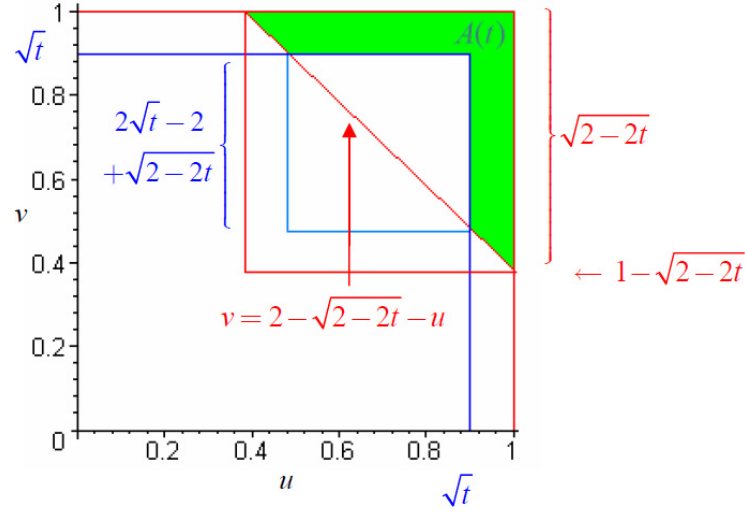
$$v_1 = v_{lower} = \begin{cases} \frac{u}{2}, & \text{if } u \leq \frac{1}{2} \\ \left(1 - \sqrt{\frac{1-u}{2}}\right)^2, & \text{otherwise} \end{cases}$$

$$\text{and } v_2 = v_{upper} = \begin{cases} 2u, & \text{if } u \leq \frac{1}{2} \\ 1, & \text{if } u > \frac{1}{2} \end{cases},$$

The lower bound is reached if  $V_1$  and  $V_2$  are close to each other, while the upper bound is reached if one of  $V_1$  or  $V_2$  is close to zero.

The subsequent graph explains our arguments for the calculation of the coefficient  $\lambda_U$  of upper tail dependence, which is given by  $\lambda_U = 0$ .

We start with some preliminary inequalities.



We have, for  $t > \frac{1}{2}$ , with  $\mu$  denoting Lebesgue measure,

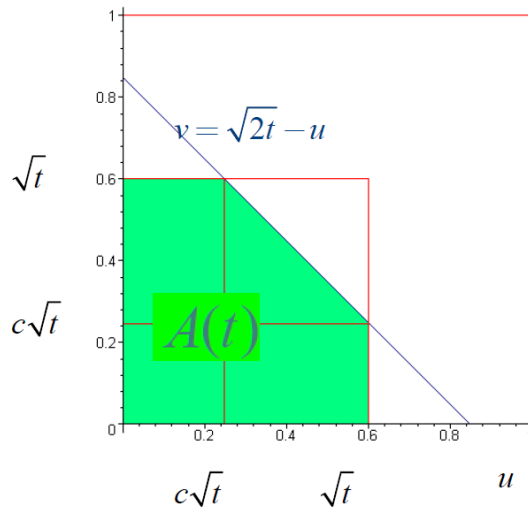
$$\begin{aligned} P(W_1 > F_1^{-1}(t), W_2 > F_2^{-1}(t)) &= P(\max(U_1, U_2) > \sqrt{t}, U_2 > 2 - \sqrt{2-t} - U_1) \\ &= \mu(A(t)) = \frac{1}{2} \left( \sqrt{2-2t}^2 - (2\sqrt{t} - 2 + \sqrt{2-2t})^2 \right) \end{aligned}$$

or, by a Taylor expansion around the point  $t = 1$ ,

$$\mu(A(t)) = \sqrt{2}(1-t)^{3/2} + \mathcal{O}((1-t)^2), \text{ hence } \lambda_U = \lim_{t \rightarrow 1} \frac{\mu(A(t))}{1-t} = 0, \text{ as stated.}$$

The subsequent graph explains our arguments for the calculation of the coefficient  $\lambda_L$  of lower tail dependence, which is given by  $\lambda_L = 2(\sqrt{2}-1) = 0,828427\dots$

We start again with some preliminary inequalities.



We have, for  $t > \frac{1}{2}$ , with  $c = \sqrt{2} - 1$ ,

$$\begin{aligned} P(\max(U_1, U_2) \leq F_2^{-1}(t), U_1 + U_2 \leq F_1^{-1}(t)) &= P(\max(U_1, U_2) \leq \sqrt{t}, U_2 \leq \sqrt{2t} - U_1) \\ &= \mu(A(t)) = t - \frac{\{(1-c)\sqrt{t}\}^2}{2} = t \left(1 - \frac{(1-c)^2}{2}\right) = 2ct \end{aligned}$$

and hence  $\lambda_L = \lim_{t \downarrow 0} \frac{\mu(A(t))}{t} = 2c = 2(\sqrt{2} - 1) = 0,828427\dots$ , as stated.

**Case 2.** Here we consider the choice

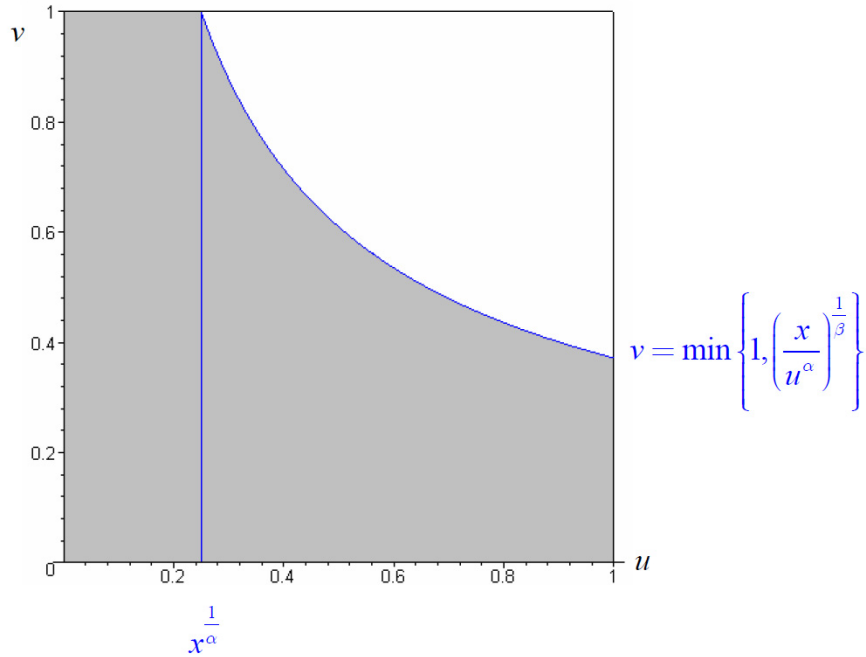
$$W_1 = T_1(U_1, U_2) = U_1^\alpha U_2^\beta, \quad W_2 = T_2(U_1, U_2) = U_1^\beta U_2^\alpha \quad \text{with real } \alpha, \beta > 0.$$

For simplicity, let  $U := U_1$ ,  $V := U_2$ .

The common c.d.f. of  $U$  and  $V$  is given by

$$F(x) = \frac{\alpha}{\alpha - \beta} x^{\frac{1}{\alpha}} - \frac{\beta}{\alpha - \beta} x^{\frac{1}{\beta}}, \quad 0 < x < 1$$

which can be seen as follows.

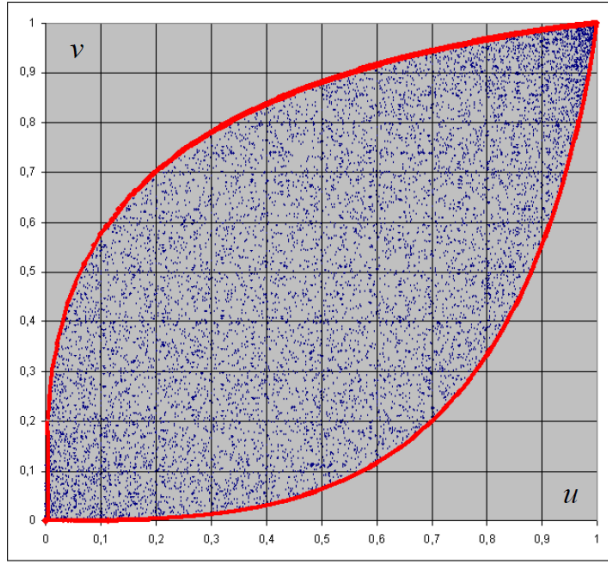


For  $0 < x < 1$ , there holds

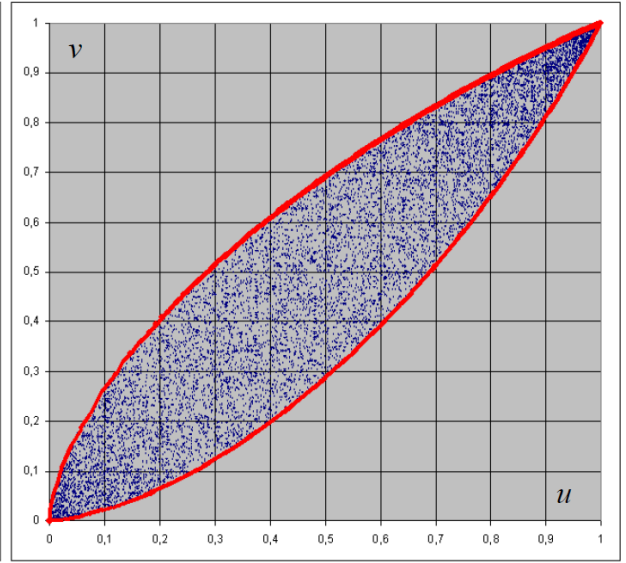
$$\begin{aligned} F(x) &= P(U^\alpha V^\beta \leq x) = P\left(V \leq \left(\frac{x}{U^\alpha}\right)^{\frac{1}{\beta}}\right) = \int_0^1 P\left(V \leq \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}}\right) du = \int_0^1 \min\left\{1, \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}}\right\} du \\ &= x^{\frac{1}{\alpha}} + x^{\frac{1}{\beta}} \int_{x^{\frac{1}{\alpha}}}^1 \frac{1}{u^{\frac{\alpha}{\beta}}} du = x^{\frac{1}{\alpha}} + \frac{\beta}{\beta - \alpha} x^{\frac{1}{\beta}} \left[1 - x^{\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)}\right] = \frac{\alpha}{\alpha - \beta} x^{\frac{1}{\alpha}} - \frac{\beta}{\alpha - \beta} x^{\frac{1}{\beta}} = P(U^\beta V^\alpha \leq x) \end{aligned}$$

by symmetry reasons.

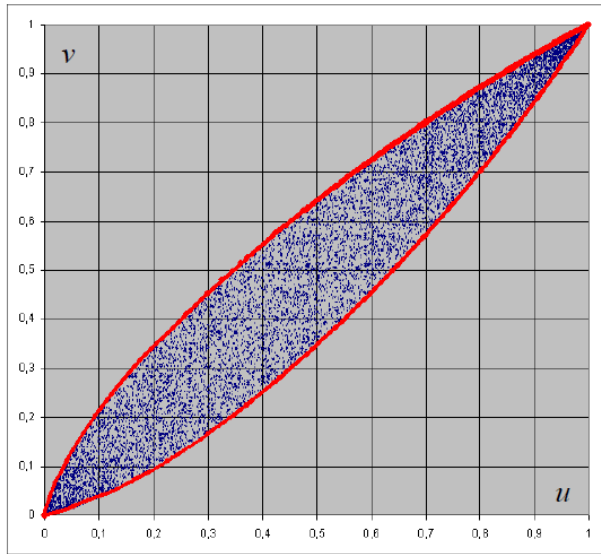
The following graphs show 10,000 simulations each of the copula given by  $(F(W_1), F(W_2))$ , for different values of  $\alpha$  and  $\beta$ .



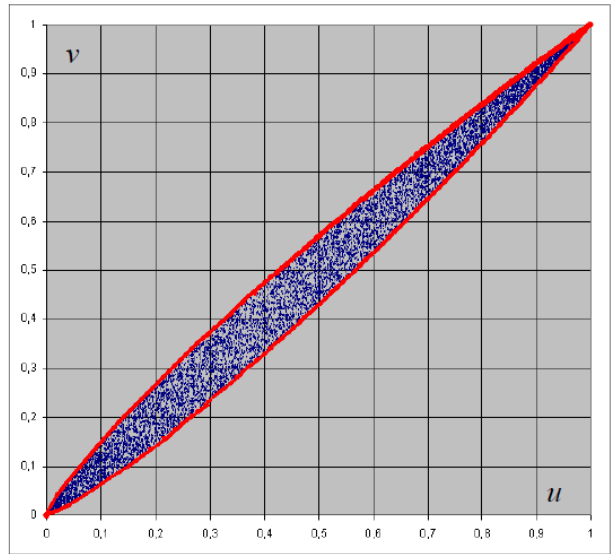
$\alpha = 3, \beta = 1$



$\alpha = 3, \beta = 2$



$\alpha = 4, \beta = 3$



$\alpha = 4, \beta = 3.5$

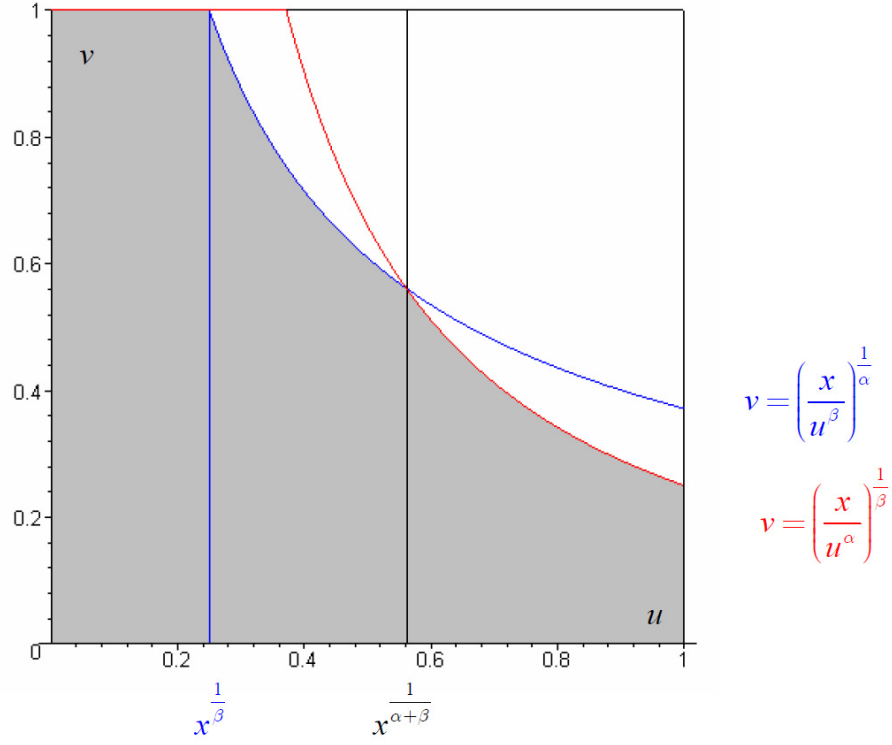
The red lines  $(u, v)$  represent the (sharp) lower and upper envelopes of the copula, which are given by

$$v_{lower} = F\left(u^{\frac{\beta}{\alpha}}\right) \text{ and } v_{upper} = F\left(u^{\frac{\alpha}{\beta}}\right), \quad 0 < u < 1.$$

Note that for  $\alpha \gg \beta$ , the copula tends to the independence copula, and for  $\alpha \approx \beta$ , we obtain the upper Fréchet bound. Note also that the copula is symmetric in  $\alpha, \beta$ .

The subsequent graph explains our arguments for the calculation of the coefficient  $\lambda_L$  of lower tail dependence, which is given by  $\lambda_L = 0$ . First notice that for  $0 < x < 1$ , the intersection point of

$\left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}}$  and  $\left(\frac{x}{u^\beta}\right)^{\frac{1}{\alpha}}$  is given by  $u_x = x^{\frac{1}{\alpha+\beta}}$  with value  $u_x$ .



Next we have

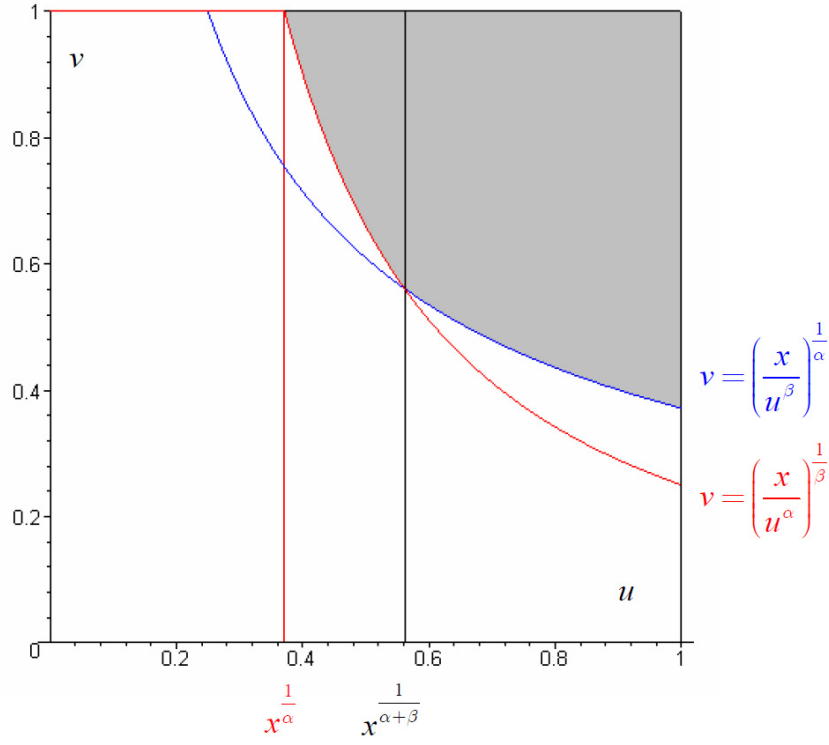
$$P(U^\alpha V^\beta \leq x, U^\beta V^\alpha \leq x) = x^{\frac{1}{\beta}} + \int_{x^{\frac{1}{\beta}}}^{x^{\frac{1}{\alpha+\beta}}} \left(\frac{x}{u^\beta}\right)^{\frac{1}{\alpha}} du + \int_{x^{\frac{1}{\alpha+\beta}}}^1 \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}} du = x^{\frac{2}{\alpha+\beta}}$$

with

$$\frac{F(x)}{P(U^\alpha V^\beta \leq x, U^\beta V^\alpha \leq x)} = \frac{\alpha}{\alpha - \beta} x^{\frac{\beta - \alpha}{\alpha(\alpha + \beta)}} - \frac{\beta}{\alpha - \beta} x^{\frac{\alpha - \beta}{\alpha(\alpha + \beta)}} \text{ and hence}$$

$$\lim_{x \rightarrow 0} \frac{F(x)}{P(U^\alpha V^\beta \leq x, U^\beta V^\alpha \leq x)} = \infty, \text{ i.e. } \lim_{x \rightarrow 0} \frac{P(U^\alpha V^\beta \leq x, U^\beta V^\alpha \leq x)}{F(x)} = 0 = \lambda_L.$$

The subsequent graph explains our arguments for the calculation of the coefficient  $\lambda_U$  of upper tail dependence, which is given by  $\lambda_U = \frac{2\beta}{\alpha + \beta} > 0$  if  $\alpha > \beta$ .



If  $\alpha > \beta$ ,

$$P(U^\alpha V^\beta > x, U^\beta V^\alpha > x) = 1 - x^{\frac{1}{\alpha}} - \int_{\frac{1}{x^\alpha}}^{\frac{1}{x^{\alpha+\beta}}} \left(\frac{x}{u^\alpha}\right)^{\frac{1}{\beta}} du - \int_{\frac{1}{x^{\alpha+\beta}}}^1 \left(\frac{x}{u^\beta}\right)^{\frac{1}{\alpha}} du = 1 - \frac{2\alpha}{\alpha - \beta} x^{\frac{1}{\alpha}} + \frac{\beta}{\alpha - \beta} x^{\frac{2}{\alpha+\beta}}$$

With, by a Taylor expansion around  $x = 1$ ,

$$\frac{P(U^\alpha V^\beta > x, U^\beta V^\alpha > x)}{1 - F(x)} = \frac{2\beta}{\alpha + \beta} - \frac{2(\alpha - \beta)}{3(\alpha + \beta)^2} (x - 1) + \mathcal{O}((x - 1)^2), \text{ hence } \lambda_U = \frac{2\beta}{\alpha + \beta} > 0.$$

## References

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