On a new class of distributions

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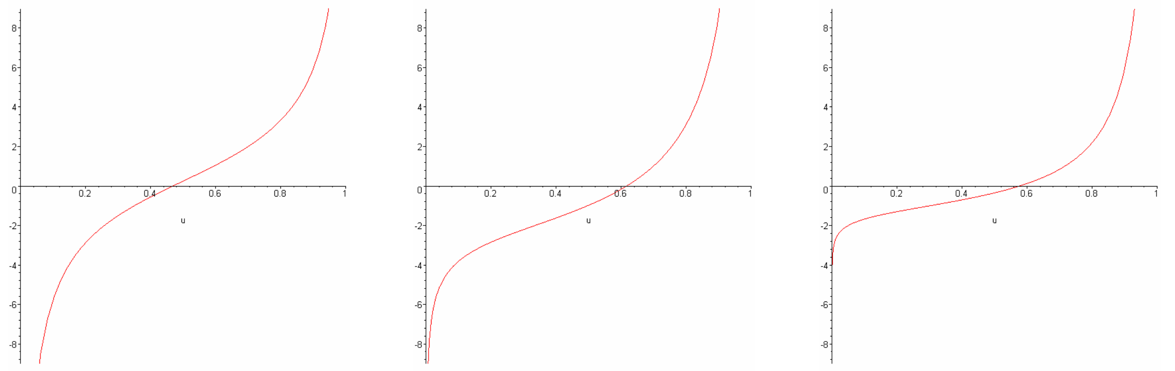
A good classical source for continuous univariate distributions is [1] and [2]. Here, in a first step, we consider continuous univariate distributions with quantile functions given by

$$Q(u;\mathbf{p}) := \frac{a}{(1-u)^b} - \frac{c}{u^d} + e \ \text{ for } \ 0 < u < 1; \ a,c > 0, \ 0 < b,d < 1, \ e \in \mathbb{R},$$

denoting $\mathbf{p} = (a,b,c,d,e)$.



$a=4,\ b=\frac{2}{5},\ c=2,\ d=\frac{3}{5},\ e=-2$ $\qquad$ $a=2,\ b=\frac{4}{5},\ c=2,\ d=\frac{3}{10},\ e=-2$ $\qquad$ $a=1,\ b=\frac{9}{10},\ c=3,\ d=\frac{1}{10},\ e=1$

examples of quantile functions for different parameter values

The advantage of these distributions is their easy simulation with standard uniform random numbers and the fact that moments – also for order statistics – can be easily calculated.

In particular, we have, for independent random variables $X_1, \cdots, X_n$, distributed as X with the above quantile function:

$$E(X;\mathbf{p}) = \int_0^1 Q(u;\mathbf{p})\,du = \int_0^1 \frac{a}{(1-u)^b} - \frac{c}{u^d} + e\,du = \frac{a}{1-b} - \frac{c}{1-d} + e$$

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$$E(X^2;\mathbf{p}) = \int_0^1 Q^2(u;\mathbf{p})\,du = \int_0^1 \left(\frac{a}{(1-u)^b} - \frac{c}{u^d} + e\right)^2 du$$

$$= \frac{a^2}{1-2b} + \frac{c^2}{1-2d} + \frac{2ae}{1-b} - \frac{2ce}{1-d} - 2ac\frac{\Gamma(1-b)\Gamma(1-d)}{\Gamma(2-b-d)} + e^2$$

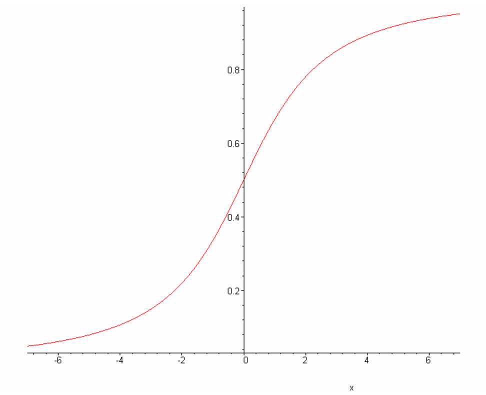
$$E\left(X_{k:n};\mathbf{p}\right) = \frac{1}{\mathrm{Beta}(k,n+1-k)}\int_0^1 Q(u,a,b,c)\,u^{k-1}(1-u)^{n-k}\,du$$

$$= \Gamma(n+1)\cdot\left[a\frac{\Gamma(n+1-b-k)}{\Gamma(n+1-k)\Gamma(n+1-b)} - c\frac{\Gamma(k-d)}{\Gamma(k)\Gamma(n+1-d)}\right] + e$$

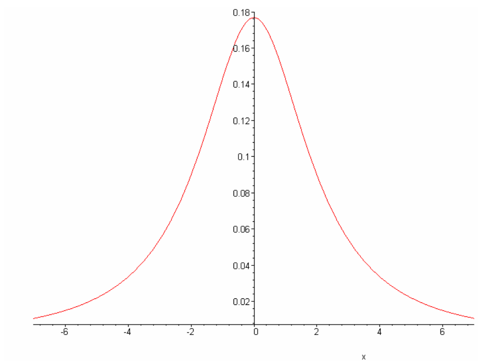
for $1 \le k \le n$.

Unfortunately, the corresponding distribution function $F(x;\mathbf{p})$ cannot be explicitly calculated in most cases. Here is a result for a special case:

$$c = a,\ b = d = \frac{1}{2},\ e = 0:\quad F(x;\mathbf{p}) = \frac{1}{2} + \frac{\mathrm{sgn}(x)}{2}\sqrt{1 - \frac{4}{\left(1+\sqrt{1+\frac{x^2}{a^2}}\right)^2}},\quad x \in \mathbb{R}$$



cdf $F(x;\mathbf{p})$

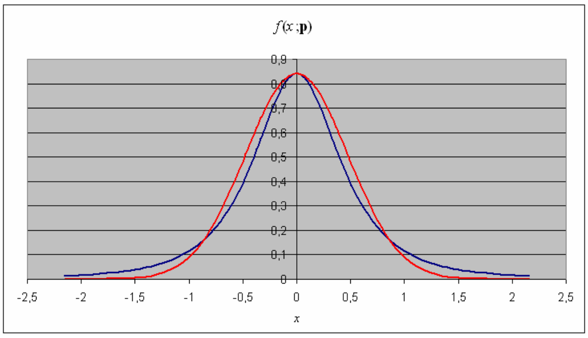


pdf $f(x;\mathbf{p}) = \frac{\partial}{\partial x}F(x;\mathbf{p})$

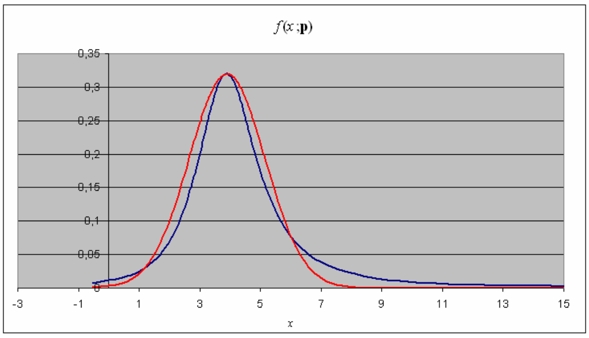
It is, however, in general possible to find an implicit relation for the density $f(x;\mathbf{p})$ by the chain rule of differentiation: from the obvious relation $F\left(Q(u;\mathbf{p});\mathbf{p}\right) = u,\ 0 < u < 1$ we get $f\left(Q(u;\mathbf{p});\mathbf{p}\right)\cdot\frac{\partial}{\partial u}Q(u,\mathbf{p}) = 1$ and hence

$$f\left(Q(u;\mathbf{p});\mathbf{p}\right) = \frac{1}{\left(\frac{\partial}{\partial u}Q(u,\mathbf{p});\mathbf{p}\right)},\quad 0 < u < 1 \text{ with } \frac{\partial}{\partial u}Q(u,\mathbf{p}) = \frac{ab}{(1-u)^{b+1}} + \frac{cd}{u^{d+1}}.$$

The following graphs show two plots for this relationship. The red line is the normal density with mean μ and standard deviation σ.



$a = c = 1 \;\; b = d = 0,25 \;\; e = 0$; $\mu = 0 \;\; \sigma = 0,475$



$a = 1 \; b = 0,6 \;\; c = 3 \;\; d = 0,2 \;\; e = 6$; $\mu = 3,9 \;\; \sigma = 1,25$

Likewise, we find the following asymptotic relations:

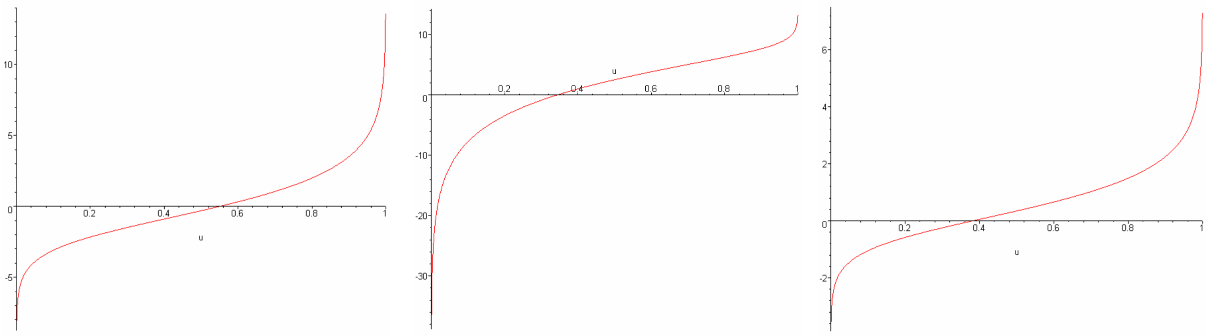
$$F(x;\mathbf{p}) \sim \begin{cases} 1 - \left(\dfrac{a}{x+c-e}\right)^{1/b}, & x \to \infty \\[2ex] \left(\dfrac{c}{a+e-x}\right)^{1/d}, & x \to -\infty. \end{cases}$$

This implies that the normalized upper extremes (maxima) are attracted to a Fréchet distribution with parameter $\alpha = \dfrac{1}{b}$; likewise the normalized lower extremes (minima) are attracted to a Weibull distribution with parameter $\alpha = \dfrac{1}{d}$.

In a second step, we consider

$Q(u;\mathbf{p}) := a\ln(u) - b\ln(1-u) + c$ for $0 < u < 1$; $a, b > 0$, $c \in \mathbb{R}$,

denoting $\mathbf{p} = (a,b,c)$.



$a = 1,\;\; b = 2,\;\; c = -1$ $\qquad$ $a = 6,\;\; b = 1,\;\; c = 6$ $\qquad$ $a = \dfrac{1}{2},\;\; b = 1,\;\; c = 0$

examples of quantile functions for different parameter values

We need the following Lemmata:

Lemma 1. We have $\int_0^1 u^m \ln(u)\, du = -\frac{1}{(m+1)^2}$ for all $m > -1$.

Proof: $\int_0^1 u^m \ln(u)\, du = -\int_0^\infty v \cdot e^{-(m+1)v}\, dv = -\frac{1}{(m+1)^2}$.

Lemma 2. We have, for integer n, k,

$$u^{k-1}(1-u)^{n-k} = \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j u^{k+j-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j (1-u)^{n+j-k} \text{ for } 0 \le u \le 1,\ 0 \le k \le n.$$

Proof. The first relation follows from the binomial expansion $(1-u)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j u^j$ for $0 \le u \le 1,\ m \in \mathbb{N}$. The second relation follows likewise from $u^m = \left(1-(1-u)\right)^m$.

From the above, we can conclude:

$$E(X;\mathbf{p}) = \int_0^1 Q(u;\mathbf{p})\, du = \int_0^1 a\ln(u) - b\ln(1-u) + c\, du = -a + b + c$$

$$E(X^2;\mathbf{p}) = \int_0^1 Q^2(u;\mathbf{p})\, du = \int_0^1 \left(a\ln(u) - b\ln(1-u) + c\right)^2 du$$
$$= 2a^2 + 2b^2 + c^2 + ab\left(\frac{\pi^2}{3} - 4\right) + 2c(b-a)$$

$$Var(X;\mathbf{p}) = a^2 + b^2 + ab\left(\frac{\pi^2}{3} - 2\right) \approx a^2 + b^2 + 1{,}2899\, ab$$

$$E(e^{tX};\mathbf{p}) = \int_0^1 \exp\left(t \cdot Q(u;\mathbf{p})\right) du = e^{ct} \int_0^1 u^{at}(1-u)^{-bt}\, du = e^{ct}\mathrm{Beta}(1+at, 1-bt),\quad -\frac{1}{a} < t < \frac{1}{b}$$

$$E\left(X_{k:n};\mathbf{p}\right) = k \cdot \binom{n}{k} \left\{ -a\sum_{j=0}^{n-k} \binom{n-k}{j} \frac{(-1)^j}{(k+j)^2} + b\sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{(n+1+j-k)^2} \right\} + c \quad \text{for } 1 \le k \le n.$$

Proof: $E\left(X_{k:n};\mathbf{p}\right) = \frac{1}{\mathrm{Beta}(k, n+1-k)} \int_0^1 Q(u,a,b,c)\, u^{k-1}(1-u)^{n-k}\, du$

with

$$\int_0^1 Q(u,a,b,c)\, u^{k-1}(1-u)^{n-k}\, du = e + a\int_0^1 \ln(u) u^{k-1}(1-u)^{n-k}\, du - b\int_0^1 \ln(1-u) u^{k-1}(1-u)^{n-k}\, du$$

and

$$\int_0^1 \ln(u)u^{k-1}(1-u)^{n-k}\,du = \sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_0^1 \ln(u)\,u^{k+j-1}\,du = -\sum_{j=0}^{n-k}\binom{n-k}{j}\frac{(-1)^j}{(k+j)^2},$$

$$\int_0^1 \ln(1-u)u^{k-1}(1-u)^{n-k}\,du = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\int_0^1 \ln(1-u)\,(1-u)^{n+j-k}\,du = -\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(-1)^j}{(n+1+j-k)^2}.$$

Note that $\mathrm{Beta}(k,n+1-k) = \dfrac{1}{k\binom{n}{k}}$.

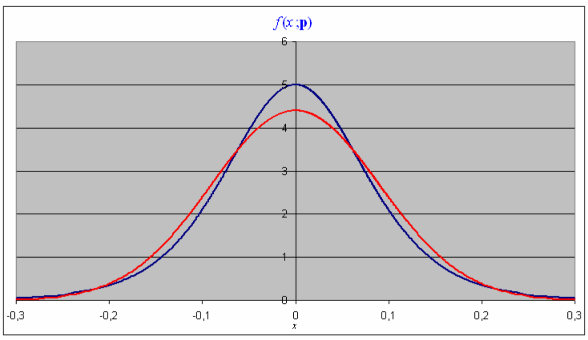
Unfortunately, as above, the corresponding distribution function $F(x;\mathbf{p})$ cannot be explicitly calculated in most cases. In the special case $a=b,$ however. we obtain a logistic distribution. For $a=2b,$ we get

$$F(x;\mathbf{p}) = \frac{1}{2}\cdot\frac{4z(x)}{\sqrt{z^2(x)+4z(x)}+z(x)} \quad \text{with } z(x)=\exp\left(\frac{x-c}{b}\right).$$

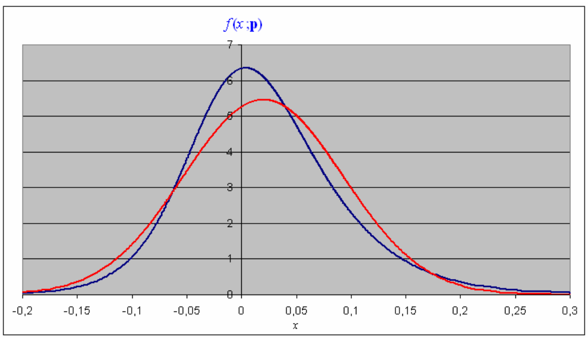
It is, however, in general possible to find an implicit relation for the density $f(x;\mathbf{p})$ by the chain rule of differentiation, as above: from the obvious relation $F\left(Q(u;\mathbf{p});\mathbf{p}\right)=u,\ 0<u<1$ we get $f\left(Q(u;\mathbf{p});\mathbf{p}\right)\cdot\dfrac{\partial}{\partial u}Q(u,\mathbf{p})=1$ and hence

$$f\left(Q(u;\mathbf{p});\mathbf{p}\right) = \frac{1}{\left(\dfrac{\partial}{\partial u}Q(u,\mathbf{p});\mathbf{p}\right)},\quad 0<u<1 \text{ with } \frac{\partial}{\partial u}Q(u,\mathbf{p}) = \frac{a}{u}+\frac{b}{1-u}.$$

The following graphs show two plots for this relationship. The red line is the normal density with identical mean μ and standard deviation σ.



$a=b=0{,}05\ \ c=0;\ \ \mu=0\ \ \sigma=0{,}09$



$a=0{,}03\ b=0{,}05\ c=0;\ \ \mu=0{,}02\ \ \sigma=0{,}073$

Examples. In [4], risk data sets originating from [3] were analyzed by means of Bernstein polynomials for empirically transformed quantile functions. Here we will compare this approach with our suggestions above. We first present the data set from [3] concerning 20 years of insured storm losses (in Mio. €):

1	2	3	4	5	6	7	8	9	10
0.468	9.951	0.866	6.731	1.421	2.040	2.967	1.200	0.426	1.946

11	12	13	14	15	16	17	18	19	20
0.676	1.184	0.960	1.972	1.549	0.819	0.063	1.280	0.824	0.227

storm losses

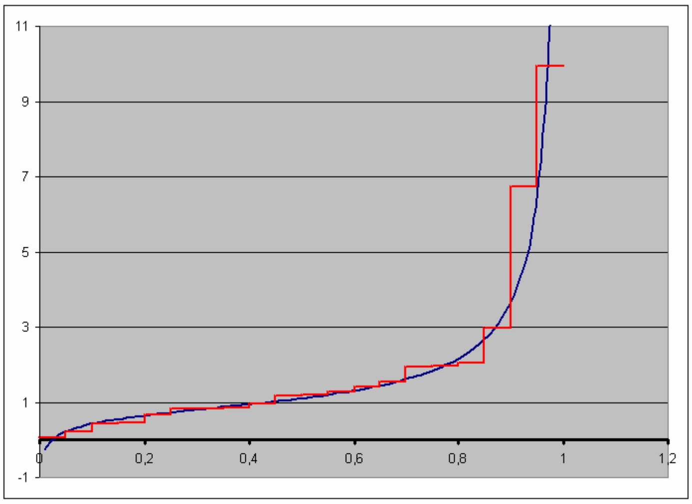
The next data set concerns 20 years of insured flooding losses (in (Mio. €):

1	2	3	4	5	6	7	8	9	10
0.966	2.679	0.897	2.249	0.956	1.141	1.707	1.008	1.065	1.162

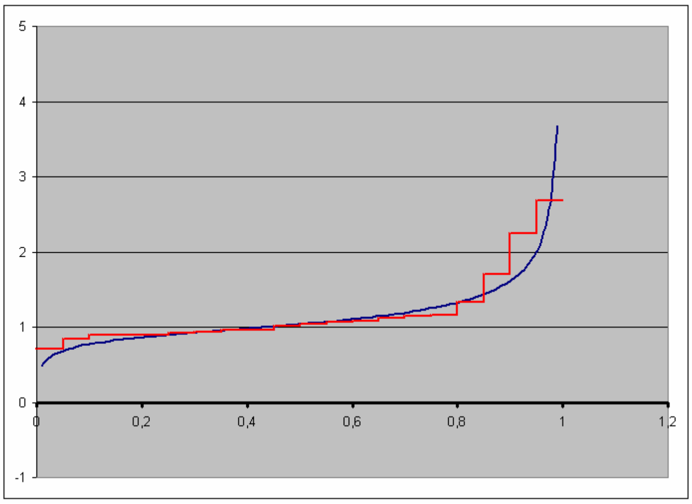
11	12	13	14	15	16	17	18	19	20
0.918	1.336	0.933	1.077	1.041	0.899	0.710	1.118	0.894	0.837

flooding losses

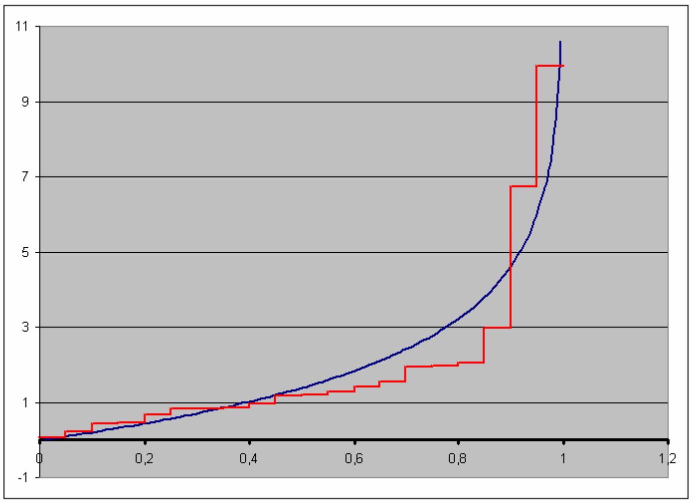
The following graphs show the manually performed adjustments of the data to the new class of quantile functions (coloured in blue):



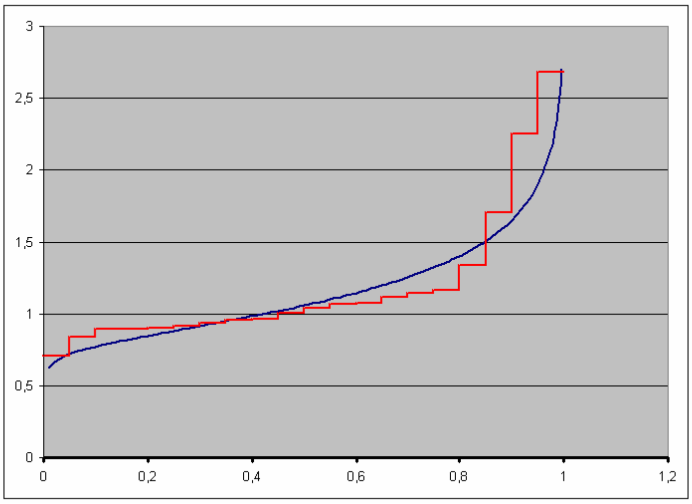
storm losses with adjusted quantile functions of type 1, with
$a = 0.4,\ b = 0.9,\ c = 2,\ d = 0.1,\ e = 2.5$



flooding losses with adjusted quantile functions of type 1, with
$a = 0.3,\ b = 0.5,\ c = 0.83,\ d = 0.1,\ e = 1.5$



storm losses with adjusted quantile functions of type 2, with $a = 0.005,\ b = 2,\ c = 0$



flooding losses with adjusted quantile functions of type 2, with $a = 0.05,\ b = 0.35,\ c = 0.85$

Seemingly, the quantile functions of type 1 fit a little better to the data than those of type 2. with our approach, we obtain the following estimates for the risk measure VaR_α for a risk level of $\alpha = 0.005$, compared to the estimates in [4]:

VaR_α	storm losses, type 1	storm losses, type 2	flooding losses, type 1	flooding losses, type 2
new approach	47.595	10.597	4.912	2.704
[4]	24.558		4.770	

References

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[3] Cottin. C. and Pfeifer. D. (2014): From Bernstein polynomials to Bernstein copulas. J. Applied Functional Analysis, vol. 9. 277 – 288

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