Математические заметки СВФУ
Апрель_июнь 2025. Том 32, № 2

Power inequalities: for which positive *a*, *b* is $a^b > b^a$?

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Abstract. In this note, we investigate the question for which positive real numbers a, b the inequality $a^b > b^a$ holds true in general.

Key words: power inequalities, Lambert W function

1. Motivation

During the first term of my mathematics study we were given the following exercise: Decide without numerical calculation which number is larger, e^π or π^e, where $e = \exp(1)$? Here is a simple approach to a solution:

2. Theorem 1

For any real number $x \geq 0$ there holds $e^x \geq x^e$ with equality only if $x = e$.

Proof: It is an elementary fact that for any real $z \neq e$, there holds $e^z \geq 1 + z$ with equality only for $z = 0$. (C.f. e.g. [1], Problem 21, p.298 or [3], Exercise 72, p. 363.) Clearly, $f(z) := e^z - 1 - z,\ z \in \mathbb{R}$ defines a strictly convex function due to $f''(z) = e^z > 0$, with a minimum attained in $z_0 = 0$ with $f(z_0) = 0$ because of $f'(z_0) = 0$. It follows that $e^{z-1} \geq z$ or $e^z \geq e \cdot z$ with equality only for $z = 1$. Replacing $e \cdot z$ with x we obtain $e^{x/e} \geq x$ for $x \in \mathbb{R}$ or $e^x \geq x^e$ for $x \geq 0$, with equality only for $x = e$. ■

Thus $e^\pi > \pi^e$. Numerically, we have $e^\pi = 23.14069264$, $\pi^e = 22.45915771$.

3. Theorem 2

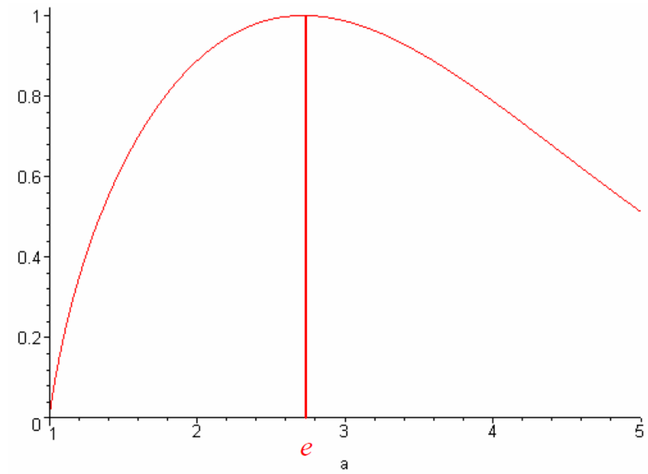
Let a be a positive real number. If $a < e$, then there holds $a^b \geq b^a$ for all $b \leq a$. If $a > e$, then there holds $a^b \geq b^a$ for all $b \geq a$. In general, we only have $a^b \geq e^a \cdot \left(\frac{\ln(a)}{a}\right)^a \cdot b^a$ and $a^b \leq e^{-b} \cdot \left(\frac{b}{\ln(b)}\right)^b \cdot b^a$ for $a, b > 0$.

Proof: By Theorem 1, the statement is true for $a = e$.

Now let $f(x,a) := \ln\left(\frac{a^x}{x^a}\right) = x\cdot\ln(a) - a\cdot\ln(x),\ x > 0.$

We have $\frac{\partial}{\partial x} f(x,a) := \ln(a) - \frac{a}{x}$ and $\frac{\partial^2}{\partial x^2} f(x,a) := \frac{a}{x^2} > 0$ for $x > 0$.

So for fixed a, $f(x,a)$ is strictly convex for $x > 0$, with $\frac{\partial}{\partial x} f(x,a) = 0$ for $x_0 := \frac{a}{\ln(a)}$ (giving a minimum point of the function in x), i.e. $f(x,a)$ is decreasing in x for $x < a \le x_0 = \frac{a}{\ln(a)}$ if $a < e$ and increasing in x for $x > a \ge x_0 = \frac{a}{\ln(a)}$ if $a < e$ with $f(a,a) = 0$ in either case. Note that $f(x_0,a) = a\cdot\left(1 - \ln(a) + \ln(\ln(a))\right) \le 0$ and equality only for $a = e$, and that by Theorem 1, $e^a \ge a^e$ with equality only for $a = e$, i.e. $a \ge e\cdot\ln(a)$ or $\ln(a) \ge 1 + \ln(\ln(a))$. This proves Theorem 2. ■



plot of the function $g(a) := e^a\cdot\left(\frac{\ln(a)}{a}\right)^a,\ a \ge 1$

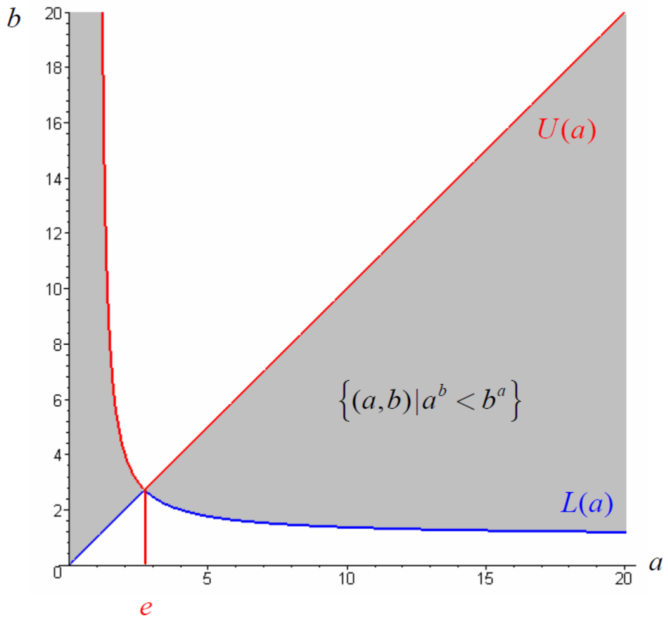
This means that the question for which positive real numbers a, b the inequality $a^b > b^a$ holds true in general can be answered as follows:

Whenever $a = e$, the inequality is true for all positive $b \ne e$. If $a \ne e$, the inequality is only partially true.

Example. Let $a = 2$ and $b = 3$. Then $a^b = 8 < 9 = b^a$. If $a = 2$ and $b = 5$, we have $a^b = 32 > 25 = b^a$. Note that by Theorem 2, we have

$$0.8875... = e^2\cdot\left(\frac{\ln(2)}{2}\right)^2 \le \frac{2^3}{3^2} = 0.\bar{8} \le e^{-3}\cdot\left(\frac{3}{\ln(3)}\right)^3 = 1.0137... \text{ and}$$

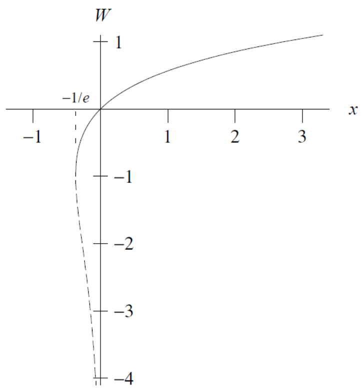
$$0.8875... = e^2 \cdot \left(\frac{\ln(2)}{2}\right)^2 \leq \frac{2^5}{5^2} = 1.28 \leq e^{-5} \cdot \left(\frac{5}{\ln(5)}\right)^5 = 1.9498...$$



plot of the complementary area $\left\{(a,b)|a^b < b^a\right\}$

Note that the lower bound $L(a)$ of this graph, colored in blue, is given by a transformation of the Lambert W function as $L(a) = \exp\left(-\mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$, $a > 0$ as can be seen as follows:

starting with the equation $a^b = b^a$, we get $b \cdot \ln(a) = a \cdot \ln(b)$. Substituting $b = e^{-c}$, this gives $e^{-c} = -\frac{a}{\ln(a)} \cdot c$, hence $c \cdot e^c = -\frac{\ln(a)}{a} = \frac{1}{a}\ln\left(\frac{1}{a}\right)$, which by inversion leads to $\mathrm{c} = \mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)$ or $b = \exp\left(-\mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$. Note further that for $0 < a < e$, we have $L(a) = a$. Likewise, it can be seen that the upper bound $U(a)$, coloured in red, is given by the expression $U(a) = \exp\left(-\mathrm{W}_{-1}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$, $a > 0$ where W_{-1} denotes the branch of W with values beneath -1. Note also that for $a > e$, we have $U(a) = a$ and $U(a) = \infty$ for $0 < a < 1$. For a thorough discussion of the Lambert W function, see [2].



graph of W_{-1} (dotted), taken from [2]

This means that we have the following final Theorem.

4. Theorem 3

For positive real numbers a,b there holds $a^b > b^a$ iff $b < L(a)$ or $b > U(a)$, with $L(a) = \exp\left(-\mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$ and $U(a) = \exp\left(-\mathrm{W}_{-1}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$ as above.

Remark. It can be shown that in general, we alternatively have $a^b \geq (\ln(a)\cdot b)^e$ with equality for $b = \dfrac{e}{\ln(a)}$. This follows from the fact that $a^b \geq e\cdot\ln(a)\cdot b$ as can be seen by a discussion of the function $f(x,a) := \ln\left(\dfrac{a^x}{e\cdot\ln(a)\cdot x}\right) = x\cdot\ln(a) - 1 - \ln(\ln(a)) - \ln(x)$ which is strictly convex in x because of $\dfrac{\partial^2}{\partial x^2} f(x,a) = \dfrac{1}{x^2} > 0$ with $\dfrac{\partial}{\partial x} f(x,a) = \ln(a) - \dfrac{1}{x} = 0$ for $x = \dfrac{1}{\ln(a)}$ and $f\left(\dfrac{1}{\ln(a)}, a\right) = 0.$

Final Remark. The topic of mathematical inequalities of different types has a long history, see e.g. [5]. Our inequalitiy is perhaps related to a paper of Seiichi Manyama [4], who proves

$a^{ea} + b^{eb} \geq a^{eb} + b^{ea}$ for all positive $a,b.$

Also, the proof of Theorem 3 gives another nice application of the Lambert W function besides those listed in [2].

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