Power inequalities: for which positive *a*, *b* is $a^b > b^a$?

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Abstract. In this note, we investigate the question for which positive real numbers *a*, *b* the inequality $a^b > b^a$ holds true in general.

Motivation. During the first term of my mathematics study we were given the following exercise: Decide without numerical calculation which number is larger, e^π or π^e, where $e = \exp(1)$? Here is a simple approach to a solution:

Theorem 1. For any real number $x \geq 0$ there holds $e^x \geq x^e$ with equality only if $x = e$.

Proof: It is an elementary fact that for any real $z \neq e$, there holds $e^z \geq 1 + z$ with equality only for $z = 0$. (C.f. e.g. [1], Problem 21, p.298 or [3], Exercise 72, p. 363.) Clearly, $f(z) := e^z - 1 - z,\ z \in \mathbb{R}$ defines a strictly convex function due to $f''(z) = e^z > 0$, with a minimum attained in $z_0 = 0$ with $f(z_0) = 0$ because of $f'(z_0) = 0$. It follows that $e^{z-1} \geq z$ or $e^z \geq e \cdot z$ with equality only for $z = 1$. Replacing $e \cdot z$ with x we obtain $e^{x/e} \geq x$ for $x \in \mathbb{R}$ or $e^x \geq x^e$ for $x \geq 0$, with equality only for $x = e$. ■

Thus $e^\pi > \pi^e$. Numerically, we have $e^\pi = 23.14069264$, $\pi^e = 22.45915771$.

Theorem 2. Let *a* be a positive real number. If $a < e$, then there holds $a^b \geq b^a$ for all $b \leq a$. If $a > e$, then there holds $a^b \geq b^a$ for all $b \geq a$. In general, we only have $a^b \geq e^a \cdot \left(\dfrac{\ln(a)}{a}\right)^a \cdot b^a$ and $a^b \leq e^{-b} \cdot \left(\dfrac{b}{\ln(b)}\right)^b \cdot b^a$ for $a, b > 0$.

Proof: By Theorem 1, the statement is true for $a = e$.

Now let $f(x,a) := \ln\left(\dfrac{a^x}{x^a}\right) = x \cdot \ln(a) - a \cdot \ln(x),\ x > 0.$

We have $\dfrac{\partial}{\partial x} f(x,a) := \ln(a) - \dfrac{a}{x}$ and $\dfrac{\partial^2}{\partial x^2} f(x,a) := \dfrac{a}{x^2} > 0$ for $x > 0$.

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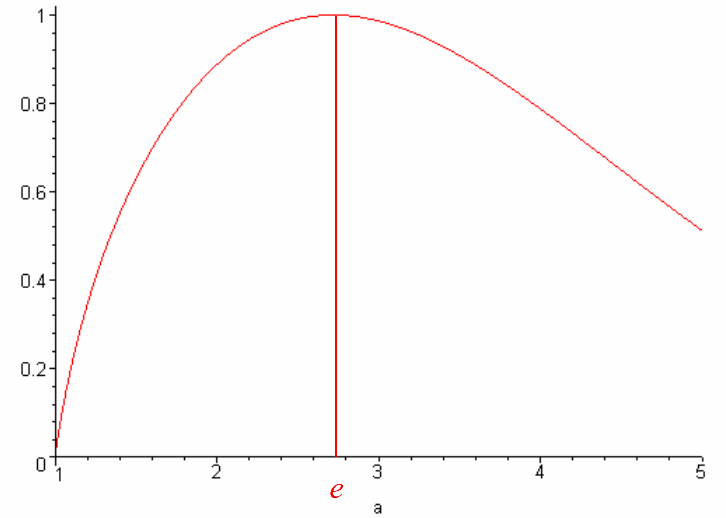
So for fixed a, $f(x,a)$ is strictly convex for $x>0$, with $\frac{\partial}{\partial x}f(x,a)=0$ for $x_0 := \frac{a}{\ln(a)}$ (giving a minimum point of the function in x), i.e. $f(x,a)$ is decreasing in x for $x<a\le x_0=\frac{a}{\ln(a)}$ if $a<e$ and increasing in x for $x>a\ge x_0=\frac{a}{\ln(a)}$ if $a<e$ with $f(a,a)=0$ in either case. Note that $f(x_0,a)=a\cdot\left(1-\ln(a)+\ln(\ln(a))\right)\le 0$ and equality only for $a=e$, and that by Theorem 1, $e^a\ge a^e$ with equality only for $a=e$, i.e. $a\ge e\cdot\ln(a)$ or $\ln(a)\ge 1+\ln(\ln(a))$. This proves Theorem 2. ■



plot of the function $g(a):=e^a\cdot\left(\frac{\ln(a)}{a}\right)^a$, $a\ge 1$

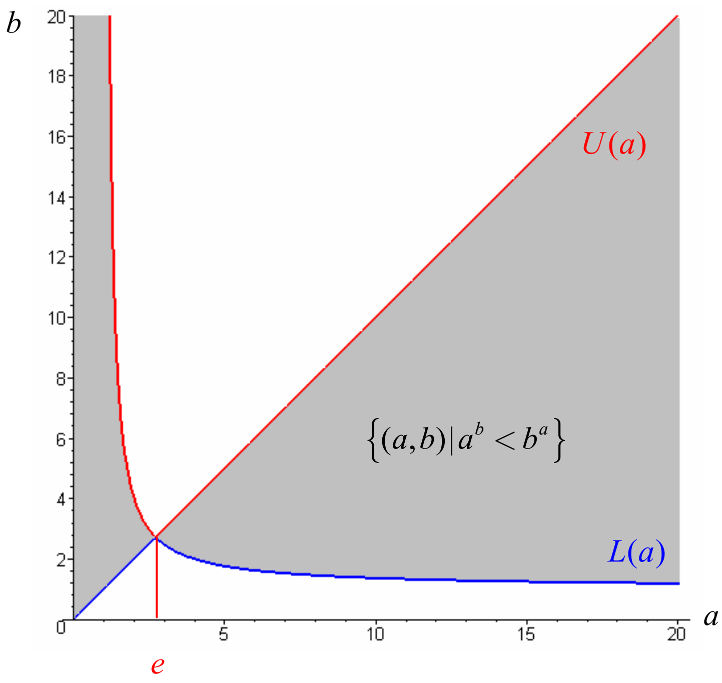
This means that the question for which positive real numbers a, b the inequality $a^b>b^a$ holds true in general can be answered as follows:

Whenever $a=e$, the inequality is true for all positive $b\ne e$. If $a\ne e$, the inequality is only partially true.

Example. Let $a=2$ and $b=3$. Then $a^b=8<9=b^a$. If $a=2$ and $b=5$, we have $a^b=32>25=b^a$. Note that by Theorem 2, we have

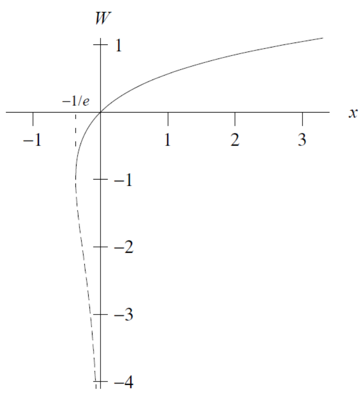
$0.8875...=e^2\cdot\left(\frac{\ln(2)}{2}\right)^2\le\frac{2^3}{3^2}=0.\bar{8}\le e^{-3}\cdot\left(\frac{3}{\ln(3)}\right)^3=1.0137...$ and

$0.8875...=e^2\cdot\left(\frac{\ln(2)}{2}\right)^2\le\frac{2^5}{5^2}=1.28\le e^{-5}\cdot\left(\frac{5}{\ln(5)}\right)^5=1.9498...$



plot of the complementary area $\{(a,b) | a^b < b^a\}$

Note that the lower bound $L(a)$ of this graph, colored in blue, is given by a transformation of the Lambert W function as $L(a) = \exp\left(-\mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$, $a > 0$ as can be seen as follows: starting with the equation $a^b = b^a$, we get $b \cdot \ln(a) = a \cdot \ln(b)$. Substituting $b = e^{-c}$, this gives $e^{-c} = -\frac{a}{\ln(a)} \cdot c$, hence $c \cdot e^c = -\frac{\ln(a)}{a} = \frac{1}{a}\ln\left(\frac{1}{a}\right)$, which by inversion leads to $\mathrm{c} = \mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)$ or $b = \exp\left(-\mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$. Note further that for $0 < a < e$, we have $L(a) = a$. Likewise, it can be seen that the upper bound $U(a)$, coloured in red, is given by the expression $U(a) = \exp\left(-\mathrm{W}_{-1}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$, $a > 0$ where W_{-1} denotes the branch of W with values beneath -1. Note also that for $a > e$, we have $U(a) = a$ and $U(a) = \infty$ for $0 < a < 1$. For a thorough discussion of the Lambert W function, see [2].



graph of W_{-1} (dotted), taken from [2]

This means that we have the following final Theorem.

Theorem 3. For positive real numbers a,b there holds $a^b > b^a$ iff $b < L(a)$ or $b > U(a)$, with $L(a) = \exp\left(-\mathrm{W}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$ and $U(a) = \exp\left(-\mathrm{W}_{-1}\left(\frac{1}{a}\ln\left(\frac{1}{a}\right)\right)\right)$ as above.

Remark. It can be shown that in general, we alternatively have $a^b \geq (\ln(a)\cdot b)^e$ with equality for $b = \frac{e}{\ln(a)}$. This follows from the fact that $a^b \geq e\cdot\ln(a)\cdot b$ as can be seen by a discussion of the function $f(x,a) := \ln\left(\frac{a^x}{e\cdot\ln(a)\cdot x}\right) = x\cdot\ln(a) - 1 - \ln(\ln(a)) - \ln(x)$ which is strictly convex in x because of $\frac{\partial^2}{\partial x^2} f(x,a) = \frac{1}{x^2} > 0$ with $\frac{\partial}{\partial x} f(x,a) = \ln(a) - \frac{1}{x} = 0$ for $x = \frac{1}{\ln(a)}$ and $f\left(\frac{1}{\ln(a)}, a\right) = 0.$

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