Some reflections on singular mixture copulas

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April 2025

Key words: alternative copula construction
2020 Mathematics Subject Classification: 062H05

In this paper, we reflect the origins of singular mixture copulas, going back to statistical considerations of sea water levels in 2009, see [4].
More elaborate treatments of singular mixture copulas can be fond in [2] and [3], including discussions of tail dependence.

Consider the following parameterized distribution function F :

$$F(x;a,b)=\begin{cases}\dfrac{b}{a}x, & 0\leq x\leq a\\[2mm] \dfrac{1-b}{1-a}(x-a)+b, & a\leq x\leq 1\end{cases}\quad \text{for } 0\leq x\leq 1\ \ (0<a,b<1).$$

The following pictures show different graphs of this function for three values of a and $b\in\left\{\dfrac{k}{10}\middle|1\leq k\leq 9\right\}$:

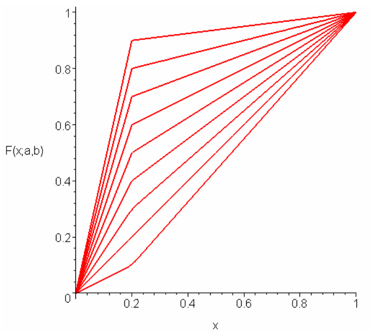

Fig.1
$a=0.2$

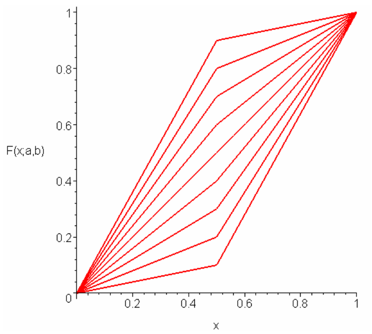

Fig.2
$a=0.5$

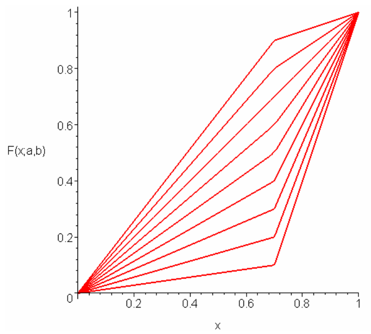

Fig.3
$a=0.7$

As can immediately be seen, the distribution function F of the continuous uniform distribution over $[0,1]$ can be obtained as a suitable mixture of such distribution functions:

$$F(x)=x=\alpha F(x;a,b)+(1-\alpha)F(x;a,c) \text{ with } \alpha=\frac{a-c}{b-c} \text{ for } 0<c<a<b<1\ \ (0\leq x\leq 1).$$

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Now let X be a random variable with a continuous uniform distribution over $[0,1]$, and I be a random variable, independent of X, with a binomial $B(1,\alpha)$-distribution. Define the random variable Y by

$$Y := I \cdot F^{-1}(X;a,b) + (1-I)\cdot F^{-1}(X;a,c) \text{ with } \alpha = \frac{a-c}{b-c} \text{ for } 0<c<a<b<1.$$

Then, by the statement above, Y also follows a continuous uniform distribution over $[0,1]$, independent of the particular choices of a, b and c, and the distribution function of the pair (X,Y) is a certain *singular* copula, given by

$$C(x,y) = P(X \le x,\ Y \le y) = \alpha \min\left(x, F(y;a,b)\right) + (1-\alpha)\min\left(x, F(y;a,c)\right),\ 0 \le x,y \le 1$$

(cf. [3], p.166). In particular, under $0<c<a<b<1$,

$$C(x,x) = P(X \le x,\ Y \le x) = \alpha x + (1-\alpha)F(x;a,c),\ 0 \le x \le 1.$$

From here we see that this copula can have a strong dependence in both tails, given by the corresponding tail dependence coefficients, which are given by

$$\lambda_L = \lim_{x\downarrow 0}\frac{C(x,x)}{x} = \alpha + (1-\alpha)\frac{c}{a} = \frac{a^2-2ac+bc}{a(b-c)} > 0$$

and

$$\lambda_U = \lim_{x\uparrow 1}\frac{1-2x+C(x,x)}{1-x} = (1-\alpha) + \alpha\frac{1-b}{1-a} = \frac{a^2-2ab+b-c+bc}{(b-c)(1-a)} > 0.$$

The following picture shows the corresponding empirical copula for the choice $a = 0.3;\ b = 0.6;\ c = 0.2;\ \alpha = 0.25$ for 10000 samples, together with a Q-Q-plot for Y:

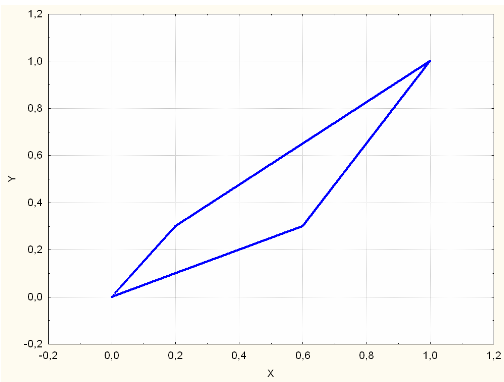

Fig.4

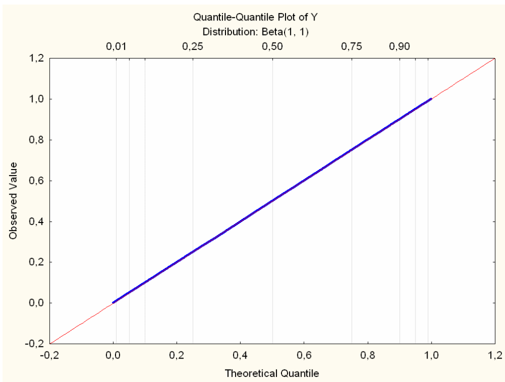

Fig.5

Here,

$$\lambda_L = \frac{a^2-2ac+bc}{a(b-c)} = \frac{3}{4} = 0.75 \text{ and } \lambda_U = \frac{a^2-2ab+b-c+bc}{(b-c)(1-a)} = \frac{25}{28} = 0.892857....$$

Note that we have, in general,

$$F^{-1}(u;a,b)=\begin{cases}\dfrac{a}{b}u, & 0\le u\le b\\ \dfrac{1-a}{1-b}(u-b)+a, & b\le u\le 1\end{cases}\quad \text{for } 0\le u\le 1.$$

A similar approach to singular copulas has been considered in [1], section 3.2 (Geometric Methods), however with a different setup related to “shuffles of M”.

The above construction gives immediately rise to various *non-singular* copulas (called *singular mixture copulas*), if we choose the parameters (a,b,c) suitably in a random way such that the side condition

$$0<c<a<b<1 \text{ (*)}$$

remains valid. Note that in this case, $\alpha=\dfrac{a-c}{b-c}$ also is a random variable in general, but dependent of (a,b,c).

Example 1. Let c follow a continuous uniform distribution over $[0,1]$, and define

$$a:=c+c\cdot(1-c)\cdot\gamma$$
$$b:=c+c\cdot(1-c)\cdot\delta$$

with fixed constants (parameters) $\gamma,\delta\in(0,1)$ such that $\gamma<\delta$. Then the side condition (*) is valid with a *non-random* $\alpha=\dfrac{\gamma}{\delta}$. The following picture shows two empirical copulas for the choices $\gamma=0.5;\ \delta=1;\ \alpha=0.5$ and $\gamma=0.4;\ \delta=0.8;\ \alpha=0.5$ for 500 samples each:

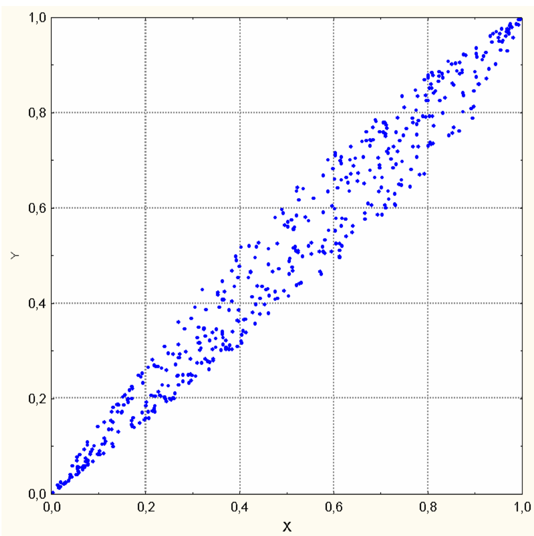

Fig.6
$\gamma=0.5;\ \delta=1;\ \alpha=0.5$

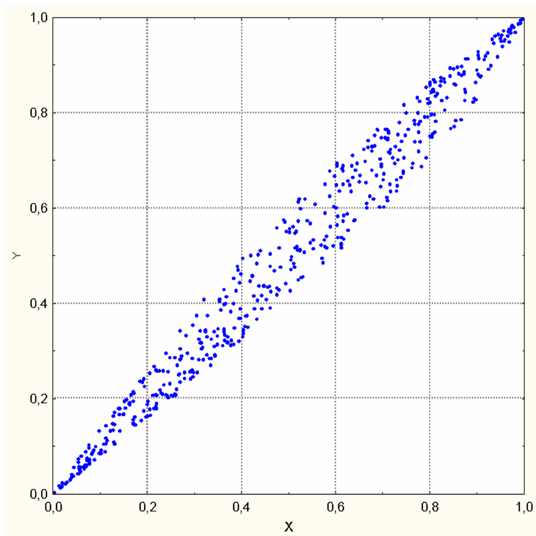

Fig.7
$\gamma=0.4;\ \delta=0.8;\ \alpha=0.5$

We can also evaluate upper and lower coefficients of tail dependence here, since by Lebesgue's theorem of dominated convergence, we can simply take expectations, i.e.

$$\lambda_L = E\left(\alpha + (1-\alpha)\frac{c}{a}\right) = \frac{\delta-\gamma}{\delta\gamma}\ln(1+\gamma) + \frac{\gamma}{\delta}$$

and

$$\lambda_U = E\left((1-\alpha) + \alpha\frac{1-b}{1-a}\right) = \frac{\delta-\gamma}{\delta\gamma}\ln(1-\gamma) + 2 - \frac{\gamma}{\delta}.$$

In our particular cases, we obtain

$$\lambda_L = \frac{\delta-\gamma}{\delta\gamma}\ln(1+\gamma) + \frac{\gamma}{\delta} \quad = \begin{cases} 0.905465\ldots & \gamma = 0.5,\ \delta = 1 \\ 0.920590\ldots & \gamma = 0.4,\ \delta = 0.8 \end{cases}$$

$$\lambda_U = \frac{\delta-\gamma}{\delta\gamma}\ln(1-\gamma) + 2 - \frac{\gamma}{\delta} = \begin{cases} 0.806852\ldots & \gamma = 0.5,\ \delta = 1 \\ 0.861467\ldots & \gamma = 0.4,\ \delta = 0.8. \end{cases}$$

Example 2. Let b follow a continuous uniform distribution over $[0,1]$, and define

$$a := b^{\gamma}$$
$$c := b^{\delta}$$

with fixed constants (parameters) $\gamma, \delta > 1$ such that $\gamma < \delta$. Then the side condition (*) is valid with a *random* $\alpha = \dfrac{b^{\delta} - b^{\gamma}}{b - b^{\gamma}}$. The following pictures show an empirical copula for the choices $\gamma = 1,5;\ \delta = 2$ for 500 samples, together with a Q-Q-plot for Y:

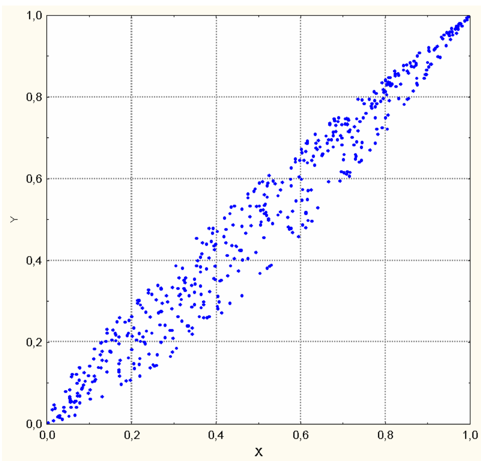

Fig.8

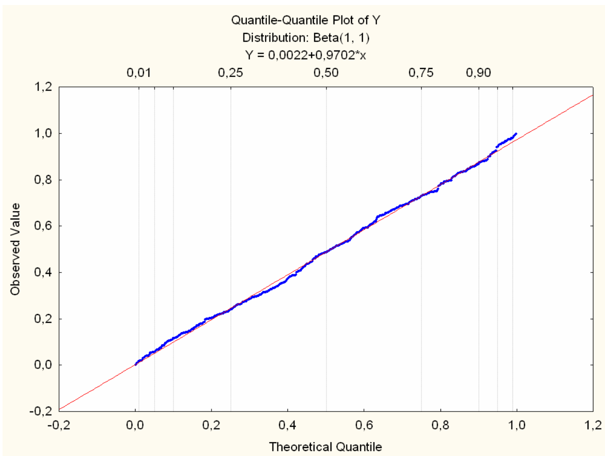

Fig.9

Note that in all example cases considered here, the support of the singular mixing copulas is a compact convex subset of the unit square, and that these copulas seem to be suitable for modelling extreme positive dependence structures in both tails.

Example 3. Let a follow a continuous uniform distribution over $[0,1]$, and define

$$c := a - \gamma a(1-a)$$
$$b := a + \gamma a(1-a)$$

with a fixed constant parameter $0 < \gamma < 1$. Then the side condition (*) is valid with $\alpha = \frac{1}{2}$. Construct the pair (X,Y) as above and, independently thereof, a $B\left(1,\frac{1}{2}\right)$-distributed random variable J and let

$$\hat{X} := JX + (1-J)Y$$
$$\hat{Y} := JY + (1-J)X$$

(symmetrization). The following picture shows an empirical copula for this construction (pairs $\left(\hat{X},\hat{Y}\right)$) with the choice $\gamma = 0,5$ for 500 samples:

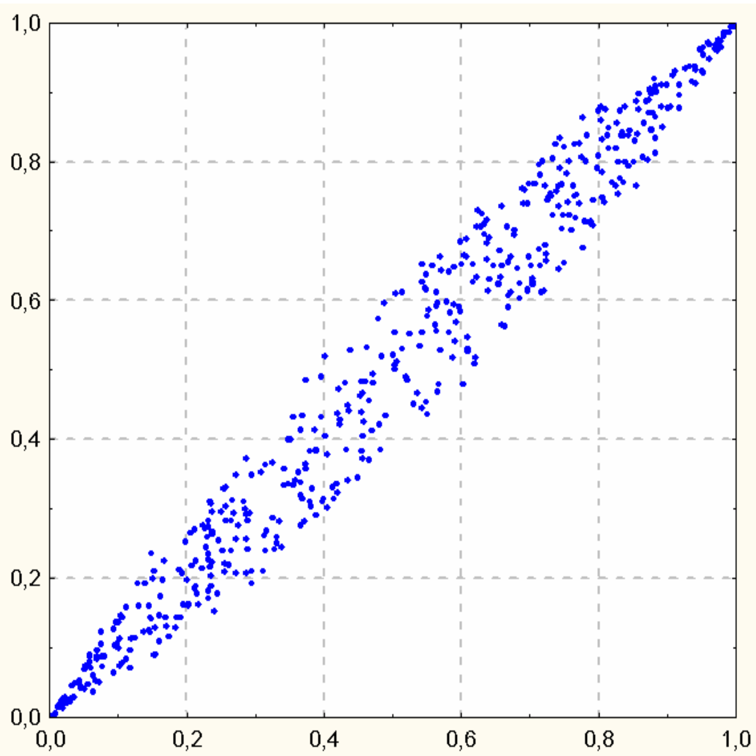

Fig.10

Final comment. This paper was motivated by a study of joint high sea water levels in the German bight during 100 years in the past, see [4]. The following picture shows an empirical copula of 500 high sea water levels from gauge stations in Cuxhaven and Helgoland in 2009, which resembles very much the pictures obtained above.

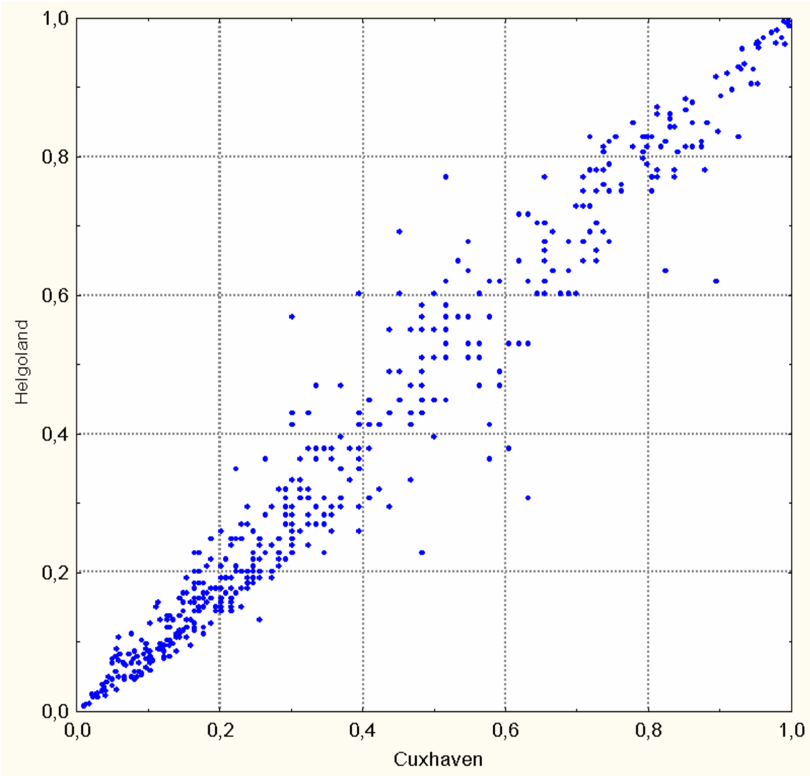

Fig.11

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