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TAIL-DEPENDENCE PROPERTIES OF SOME NEW TYPES OF COPULA MODELS

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Abstract

We investigate the tail-dependence behaviour of some new types of copula models, published recently in [8].

1. Introduction

There are many approaches to copula modelling in the literature, cf., e.g., the References below. Here we consider the following general approach: let $\mathbf{U} = \{U_k\}_{k \in \mathbb{N}}$ be a sequence of independent standard random variables, i.e., each U_k has a continuous uniform distribution over the interval $[0, 1]$. Let further $T_1, \ldots, T_n$, $n \in \mathbb{N}$ be real continuous functions over $\mathbb{R}^{\mathbb{N}}$ and $V_i = T_i(\mathbf{U})$ for $i = 1, \ldots, n$ with a continuous

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uniform distribution over $[0, 1]$ each. Then $\mathbf{V} = (V_1, \ldots, V_n)$ is a representative of an n-dimensional copula.

Note that if $W_i = T_i(\mathbf{U})$ is not directly uniformly distributed then $V_i = F_i(W_i)$ is so if F_i denotes the c.d.f. of W_i.

Of particular interest especially for financial markets or risk management is the tail dependence of copulas w.r.t. to joint extremes, see, e.g., [1] or [3]. While in [5], [6] and [7], the topic of tail dependence was explicitly treated for dependence-of-unity copulas, it was not addressed for the new approach in [8] yet, which we shall catch up on here. For the sake of simplicity, we restrict ourselves to the two-dimensional case $n = 2$ with $(W_1(\mathbf{U}), W_2(\mathbf{U}))$ representing the pre-copula construction. The simplest definition of the coefficient λ_U of upper and λ_L of lower tail dependence is

$$\lambda_U = \lim_{t \uparrow 1} \frac{P(W_1(\mathbf{U}) > F^{-1}(t),\, W_2(\mathbf{U}) > G^{-1}(t))}{1 - t},$$

$$\lambda_L = \lim_{t \downarrow 0} \frac{P(W_1(\mathbf{U}) \leq F^{-1}(t),\, W_2(\mathbf{U}) \leq G^{-1}(t))}{t},$$

where F denotes the c.d.f. of $W_1(\mathbf{U})$ and G the c.d.f. of $W_2(\mathbf{U})$, see, e.g., ([5], Def. 7.36, p. 247).

2. Particular Cases

Case 1. In [8], this refers to Case 2. Let $W_1(\mathbf{U}) = U_1 + U_2$, $W_2(\mathbf{U}) = U_1 \cdot U_2$. It is easy to see that the c.d.f. G is given by

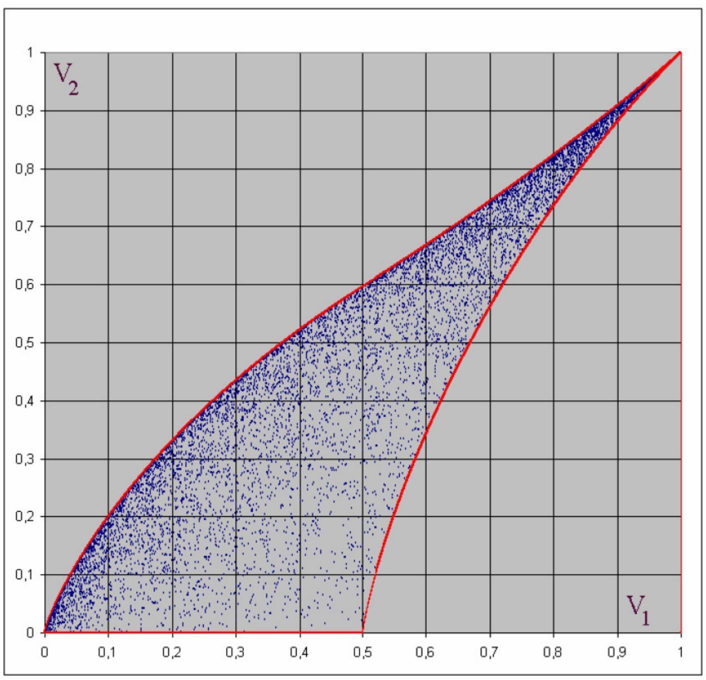
$$G(x) = (1 - \ln(x)) \cdot x, \quad 0 < x \leq 1 \text{ and}$$

$$F(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x \le 1, \\ 1 - 2\left(1 - \dfrac{x}{2}\right)^2, & 1 \le x \le 2. \end{cases}$$

The first formula follows from the observation that $-\ln(W_2(\mathbf{U}))$ represents the sum of two independent standard exponentially distributed random variables, hence is gamma-distributed. The inverse function G^{-1} is not available in elementary form, but F^{-1} can easily be calculated as

$$F^{-1}(t) = \begin{cases} \sqrt{2t}, & 0 \le t \le \dfrac{1}{2}, \\ 2 - \sqrt{2 - 2t}, & \dfrac{1}{2} \le t \le 1. \end{cases}$$

The following graph shows $10{,}000$ simulations of $\mathbf{V}$.



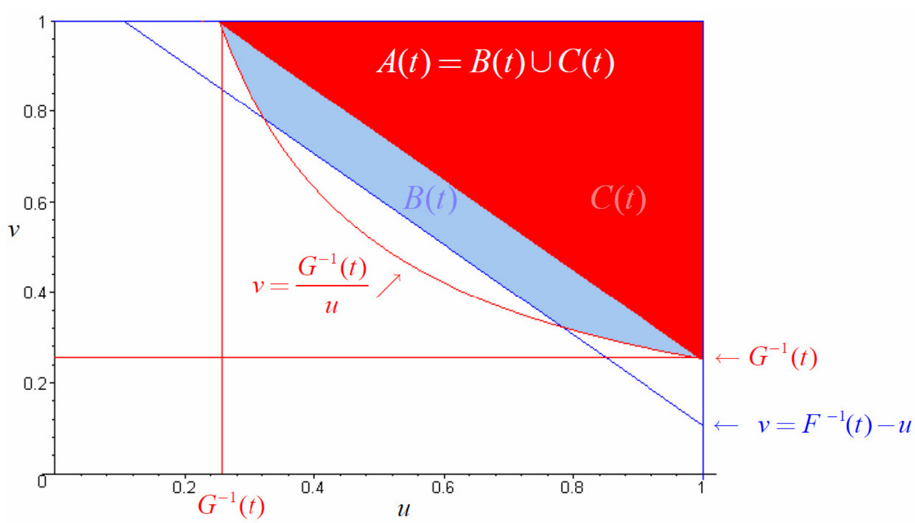
The red lines (u, v) represent the lower and upper envelopes of the copula, which are given by

$$v_{lower} = \begin{cases} 0, & \text{if } u \le \frac{1}{2}, \\ (1 - \sqrt{2 - 2u})(1 + \ln(1 - \sqrt{2 - 2u})), & \text{otherwise} \end{cases}$$

and $$v_{upper} = \begin{cases} \frac{u}{2} \cdot \left(1 - \ln\left(\frac{u}{2}\right)\right), & \text{if } u \le \frac{1}{2}, \\ \left(1 - \frac{1}{2}\sqrt{1 - u}\right) \cdot \left(1 - 2\ln\left(1 - \frac{1}{2}\sqrt{1 - u}\right)\right), & \text{if } u > \frac{1}{2}, \end{cases}$$

see [8].

In what follows we denote by μ the two-dimensional Lebesgue measure. The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 1$. We use some preliminary inequalities.



For $0 \le t \le 1$, we have

$$P\left(U+V \geq F^{-1}(t),\, U \cdot V \geq G^{-1}(t)\right) = P\left(V \geq F^{-1}(t) - U,\, V \geq \frac{G^{-1}(t)}{U}\right)$$

$$= \mu(A(t)) \geq \mu(C(t)) = \frac{\left(1 - G^{-1}(t)\right)^2}{2}$$

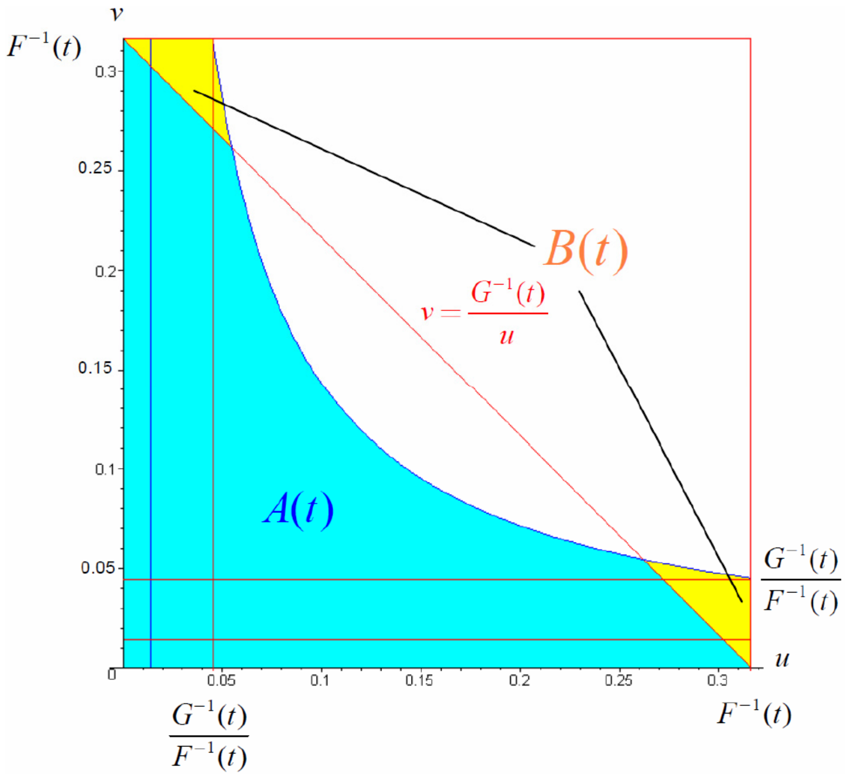
which, by the substitution $s = G^{-1}(t)$ or $t = G(s)$, gives

$$\lambda_U = \lim_{t \uparrow 1} \frac{1}{1-t} P\left(U+V \geq F^{-1}(t),\, U \cdot V \geq G^{-1}(t)\right)$$

$$\geq \lim_{t \uparrow 1} \frac{\left(1 - G^{-1}(t)\right)^2}{2(1-t)} = \lim_{s \uparrow 1} \frac{(1-s)^2}{2(1-G(s))} = 1,$$

hence $\lambda_U = 1$, as stated. Note that by a Taylor expansion around the point $s = 1$, we have $G(s) = 1 - \frac{1}{2}(s-1)^2 + \mathcal{O}\left((s-1)^3\right)$, hence $\frac{2(1-G(s))}{(1-s)^2}$ $= 1 + \mathcal{O}((s-1)$.

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 0$.



Here we use the inequality

$$\mu(A(t)) \le \mu(A(t) \cup B(t)) = \frac{G^{-1}(t)}{F^{-1}(t)} \cdot F^{-1}(t) + \int_{\frac{G^{-1}(t)}{F^{-1}(t)}}^{F^{-1}(t)} \frac{G^{-1}(t)}{u}\, du$$

$$= G^{-1}(t) + G^{-1}(t) \ln\left(\frac{F^{-1}(t)^2}{G^{-1}(t)}\right), \quad 0 < t < 1,$$

where $\mu(A(t)) = P(U + V \le F^{-1}(t),\, U \cdot V \le G^{-1}(t))$.

Now, since $F^{-1}(t) \approx \sqrt{2t}$ for $t \downarrow 0$ and $\lim_{s \downarrow 0} G(s) = 0$, with the substitution $t = G(s)$ or $s = G^{-1}(t)$,

$$\lambda_L = \lim_{t\downarrow 0}\frac{\mu(A(t))}{t} \le \lim_{t\downarrow 0}\frac{1}{t}\left[G^{-1}(t) + G^{-1}(t)\ln\left(\frac{F^{-1}(t)^2}{G^{-1}(t)}\right)\right]$$

$$= \lim_{s\downarrow 0}\frac{s}{G(s)}\left[1 + \ln\left(\frac{F^{-1}(G(s))^2}{s}\right)\right] = \lim_{s\downarrow 0}\frac{s}{G(s)}\left[1 + \ln\left(\frac{2G(s)}{s}\right)\right]$$

$$= \lim_{s\downarrow 0}\frac{1 + \ln(2)}{1 - \ln(s)} + \lim_{s\downarrow 0}\frac{\ln(1 - \ln(s))}{1 - \ln(s)} = \lim_{z\uparrow\infty}\frac{\ln(1 + z)}{1 + z} = 0$$

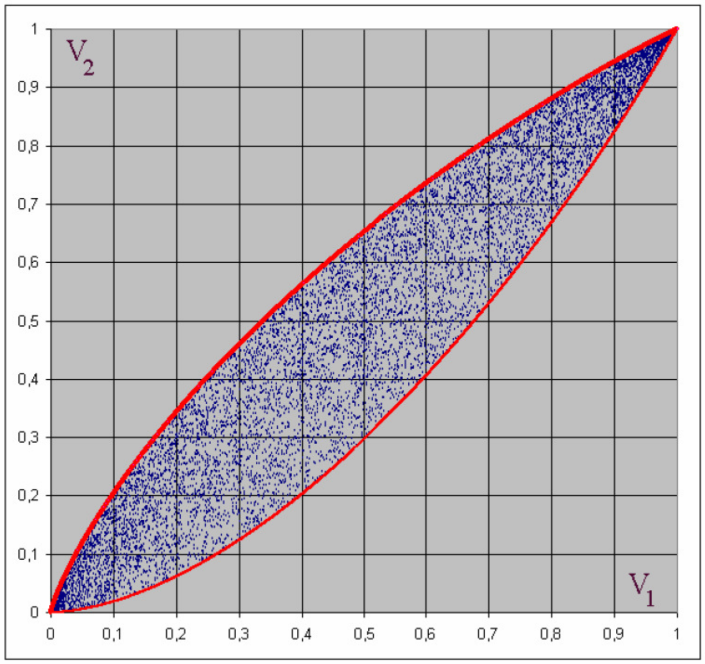
by the substitution $s = e^{-z}$, i.e., $\lambda_L = 0$, as stated.

Case 2. In [8], this refers to Case 6. We consider $W_1(\mathbf{U}) = \min(U_1, U_2)$, $W_2(\mathbf{U}) = U_1 \cdot U_2$. It is easy to see that the c.d.f. G is given by

$$G(x) = (1 - \ln(x)) \cdot x, \quad 0 < x \le 1 \text{ as above and}$$

$$F(x) = 1 - (1 - x)^2, \quad 0 \le x \le 1.$$

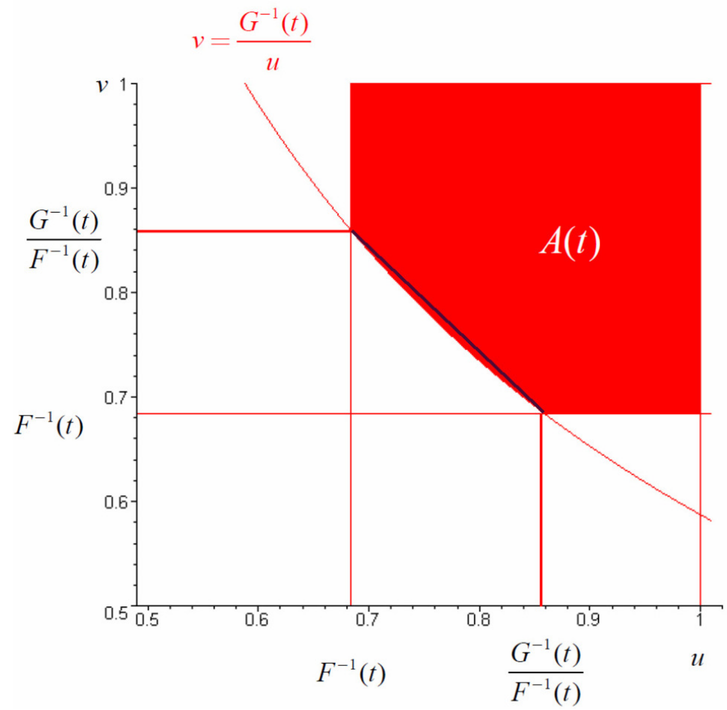
The following graph shows 10,000 simulations of $\mathbf{V}$.



The red lines (u, v) represent the lower and upper envelopes of the copula, which are given by $v_{lower} = (1 - \sqrt{1-u})^2 \cdot (1 - 2\ln(1 - \sqrt{1-u}))$ and $v_{upper} = (1 - \sqrt{1-u}) \cdot (1 - \ln(1 - \sqrt{1-u})),\ 0 < u < 1,$ see [8].

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 2 \cdot (\sqrt{2} - 1).$

We start again with some preliminary inequalities.



We have

$$P(\min(U, V) \geq F^{-1}(t),\, U \cdot V \geq G^{-1}(t))$$

$$= P\left(U \geq F^{-1}(t),\, V \geq F^{-1}(t),\, V \geq \frac{G^{-1}(t)}{U}\right)$$

$$= \mu(A(t)) \geq (1 - F^{-1}(t))^2 - \frac{\left(\frac{G^{-1}(t)}{F^{-1}(t)} - F^{-1}(t)\right)^2}{2}$$

$$= 1 - t - \frac{\left(\frac{G^{-1}(t)}{F^{-1}(t)} - F^{-1}(t)\right)^2}{2}$$

or

$$\frac{P(\min(U, V) \geq F^{-1}(t), U \cdot V \geq G^{-1}(t))}{1-t} \geq 1 - \frac{\left(\frac{G^{-1}(t)}{F^{-1}(t)} - F^{-1}(t)\right)^2}{2(1-t)},$$

hence, by the substitution $s = G^{-1}(t)$ or $t = G(s)$ as above,

$$\lambda_U \lim_{t \to 1} \frac{P(\min(U, V) \geq F^{-1}(t), U \cdot V \geq G^{-1}(t))}{1-t}$$

$$\geq = 1 - \lim_{s \to 1} \frac{\left(\frac{s}{F^{-1}(G(s))} - F^{-1}(G(s))\right)^2}{2(1 - G(s))}$$

$$= 1 - \lim_{s \to 1} \frac{K(s)}{2(1 - G(s))} = 2 \cdot (\sqrt{2} - 1) = 0.828427125\ldots$$

with $K(s) = \left(\frac{s}{F^{-1}(G(s))} - F^{-1}(G(s))\right)^2, \quad s = 0 \ldots 1.$

Note that by a Taylor expansion around the point $s = 1$, we obtain $K(s) = (\sqrt{2} - 1)^2 (s-1)^2 + \mathcal{O}((s-1)^3)$ and $2(1 - G(s)) = (s-1)^2 + \mathcal{O}((s-1)^3)$, such that $1 - \lim_{s \to 1} \frac{K(s)}{2(1 - G(s))} = 1 - (\sqrt{2} - 1)^2 = 2 \cdot (\sqrt{2} - 1)$.

On the other side, using the fact that the graph of $v = \frac{G^{-1}(t)}{u}$ is convex and thus the map $v = 2\sqrt{G^{-1}(t)} - u$ is a lower tangent at the point $u = \sqrt{G^{-1}(t)}$, we get an upper bound for the tail-dependence coefficient given by

$$\lambda_U \leq 1 - \lim_{s \to 1} \frac{(2\sqrt{G^{-1}(t)} - 2F^{-1}(t))^2}{2(1-t)} = 1 - \lim_{s \to 1} \frac{(2\sqrt{s} - 2F^{-1}(G(s)))^2}{2(1 - G(s))}$$

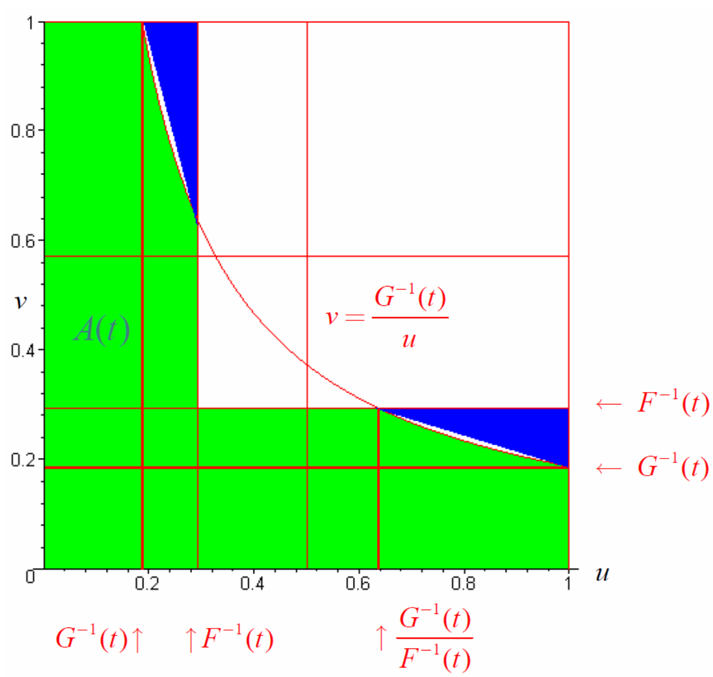
$$= 1 - \lim_{s \to 1} \frac{J(s)}{2(1 - G(s))} = 2 \cdot (\sqrt{2} - 1)$$

with $J(s) = (2\sqrt{s} - 2F^{-1}(G(s)))^2$, which gives an identical estimate as above. Note that by a Taylor expansion around the point $s = 1$, we obtain $J(s) = (\sqrt{2} - 1)^2 (s - 1)^2 + \mathcal{O}((s - 1)^3)$, hence

$$\lambda_U = 2 \cdot (\sqrt{2} - 1) = 0.828427125\ldots .$$

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 0$.

We start again with some preliminary inequalities.



We have

$$\mu(A(t)) \le 1 - (1 - F^{-1}(t))^2 - 2(F^{-1}(t) - G^{-1}(t))\left(1 - \frac{G^{-1}(t)}{F^{-1}(t)}\right)$$

$$= t - 2\,\frac{(F^{-1}(t) - G^{-1}(t))^2}{F^{-1}(t)},$$

hence

$$\lambda_L = \lim_{t\downarrow 0} \frac{P(\min(U, V) \le F^{-1}(t), U \cdot V \le G^{-1}(t))}{1-t} = \lim_{t\downarrow 0} \frac{\mu(A(t))}{t}$$

$$\le 1 - 2\lim_{t\downarrow 0} \frac{(F^{-1}(t) - G^{-1}(t))^2}{t \cdot F^{-1}(t)} = 1 - 2\lim_{s\downarrow 0} \frac{(F^{-1}(G(s)) - s)^2}{G(s) \cdot F^{-1}(G(s))} = 0.$$

Note that by a Taylor expansion of $F^{-1}(t)$ around the point $t = 0$, we get

$F^{-1}(t) = \frac{t}{2} + \mathcal{O}(t^3)$, hence $(F^{-1}(G(s)) - s)^2 \approx \left(\frac{G(s)}{2} - s\right)^2$ and thus

$$2\,\frac{(F^{-1}(G(s)) - s)^2}{G(s) \cdot F^{-1}(G(s))} = \frac{(F^{-1}(G(s)) - s)^2}{\frac{G(s)}{2} \cdot (F^{-1}(G(s))} \approx \frac{\left(\frac{G(s)}{2} - s\right)^2}{\left(\frac{G(s)}{2}\right)^2}$$

$$= \left(1 - \frac{2s}{G(s)}\right)^2 = \left(1 - \frac{2}{1 - \ln(s)}\right)^2$$

with $\lim_{s\downarrow 0}\left(1 - \frac{2}{\ln(s)}\right)^2 = 0$. This implies $\lambda_L = 0$, as stated.

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