Tail-dependence properties of some new types of copula models (extended version)

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Abstract We investigate the tail-dependence behaviour of some new types of copula models, published recently in [8].

1. Introduction. There are many approaches to copula modelling in the literature, cf. e.g. the References below. Here we consider the following general approach: let $\mathbf{U}=\{U_k\}_{k\in\mathbb{N}}$ be a sequence of independent standard random variables, i.e. each U_k has a continuous uniform distribution over the interval $[0,1]$. Let further $T_1,\cdots,T_n$, $n\in\mathbb{N}$ be real continuous functions over $\mathbb{R}^{\mathbb{N}}$ and $V_i=T_i(\mathbf{U})$ for $i=1,\cdots,n$ with a continuous uniform distribution over $[0,1]$ each. Then $\mathbf{V}=(V_1,\cdots,V_n)$ is a representative of an *n*-dimensional copula.

Note that if $W_i=T_i(\mathbf{U})$ is not directly uniformly distributed then $V_i=F_i(W_i)$ is so if F_i denotes the c.d.f. of W_i.

Of particular interest especially for financial markets or risk management is the tail dependence of copulas w.r.t. to joint extremes, see e.g. [1] or [3]. While in [5], [6] and [7], the topic of tail dependence was explicitly treated for dependence-of-unity copulas, it was not addressed for the new approach in [8] yet, which we shall catch up on here. For the sake of simplicity, we restrict ourselves to the two-dimensional case $n=2$ with $(W_1(\mathbf{U}),W_2(\mathbf{U}))$ representing the pre-copula construction. The simplest definition of the coefficient λ_U of upper and λ_L of lower tail dependence is

$$\lambda_U=\lim_{t\uparrow 1}\frac{P\left(W_1(U)>F^{-1}(t),W_2(U)>G^{-1}(t)\right)}{1-t},\quad \lambda_L=\lim_{t\downarrow 0}\frac{P\left(W_1(U)\le F^{-1}(t),W_2(U)\le G^{-1}(t)\right)}{t},$$

where *F* denotes the c.d.f. of $W_1(U)$ and *G* the c.d.f. of $W_2(U)$, see e.g. [5], Def. 7.36, p.247.

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2. Particular Cases.

Case 1. In [8], this refers to Case 2. Let $W_1(\mathbf{U}) = U_1 + U_2$, $W_2(\mathbf{U}) = U_1 \cdot U_2$. It is easy to see that the c.d.f. G is given by

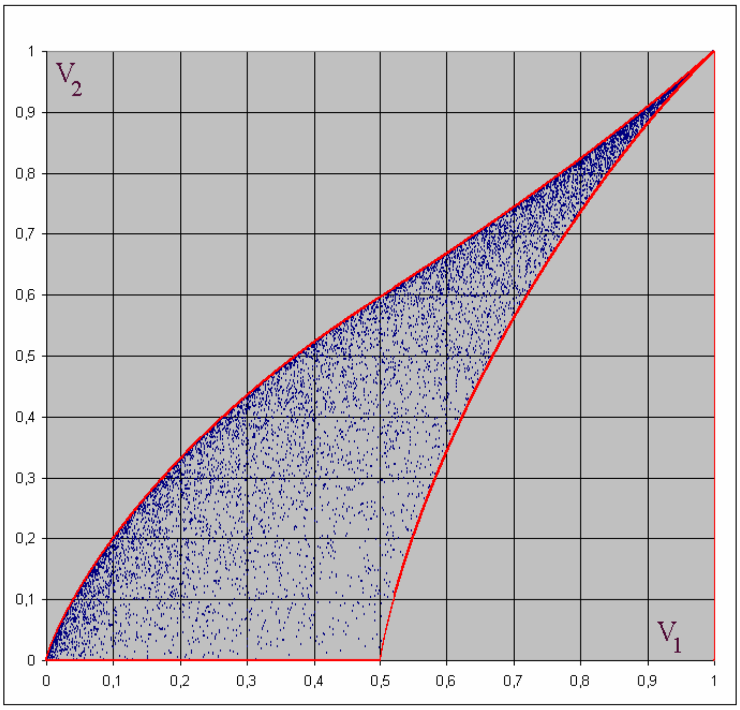
$$G(x) = \left(1 - \ln(x)\right) \cdot x, \; 0 < x \leq 1 \;\text{ and}$$

$$F(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \leq x \leq 1 \\ 1 - 2\left(1 - \dfrac{x}{2}\right)^2, & 1 \leq x \leq 2. \end{cases}$$

The first formula follows from the observation that $-\ln\left(W_2(\mathbf{U})\right)$ represents the sum of two independent standard exponentally distributed random variables, hence is gamma-distributed. The inverse function G^{-1} is not available in elementary form, but F^{-1} can easily be calculated as

$$F^{-1}(t) = \begin{cases} \sqrt{2t}, & 0 \leq t \leq \dfrac{1}{2} \\ 2 - \sqrt{2 - 2t}, & \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

The following graph shows 10.000 simulations of $\mathbf{V}$.

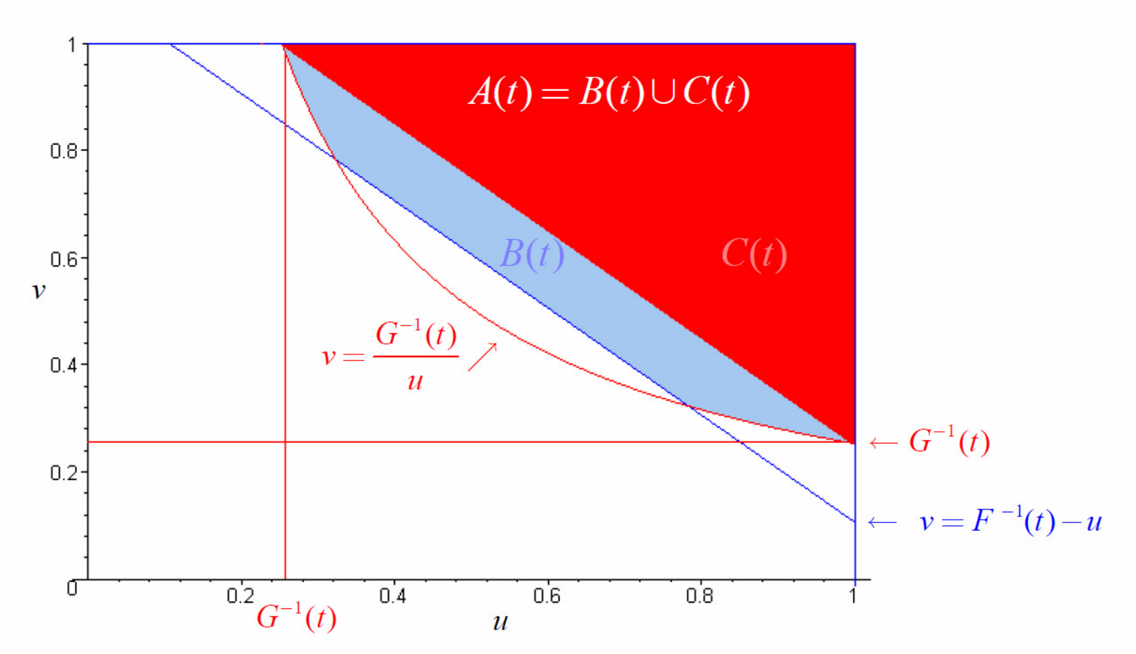


The red lines (u, v) represent the lower and upper envelopes of the copula, which are given by

$$v_{lower} = \begin{cases} 0, & \text{if } u \le \frac{1}{2} \\ \left(1-\sqrt{2-2u}\right)\left(1+\ln\left(1-\sqrt{2-2u}\right)\right), & \text{otherwise} \end{cases}$$

and $v_{upper} = \begin{cases} \frac{u}{2}\cdot\left(1-\ln\left(\frac{u}{2}\right)\right) & \text{if } u \le \frac{1}{2} \\ \left(1-\frac{1}{2}\sqrt{1-u}\right)\cdot\left(1-2\ln\left(1-\frac{1}{2}\sqrt{1-u}\right)\right), & \text{if } u > \frac{1}{2} \end{cases}$, see [8].

In what follows we denote by μ the two-dimensional Lebesgue measure. The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 1$. We use some preliminary inequalities.



For $0 \le t \le 1$, we have

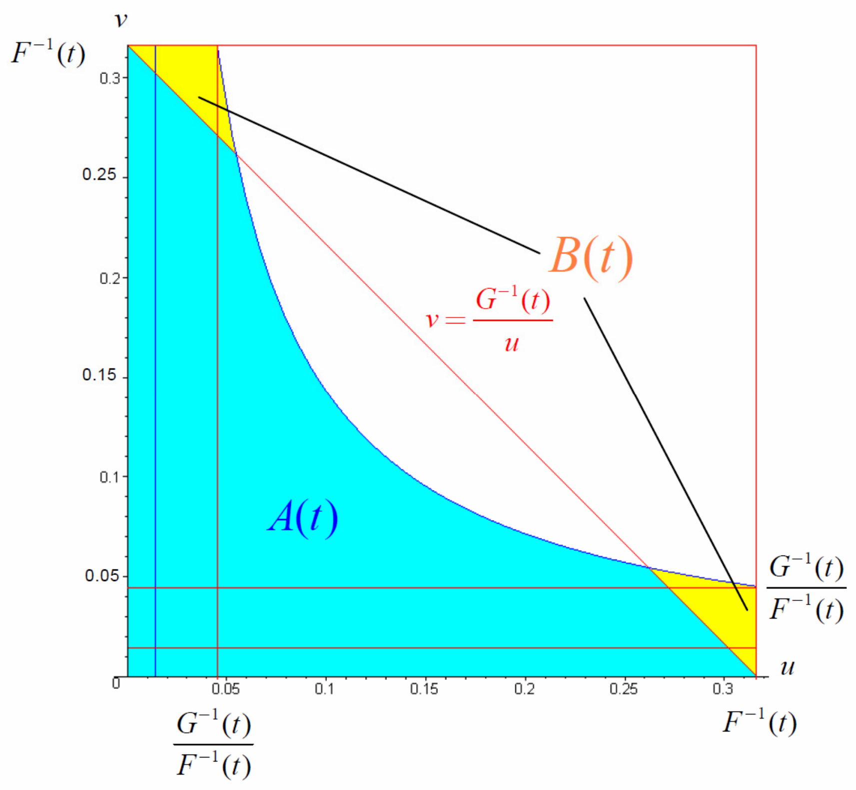
$$P\left(U+V > F^{-1}(t), U\cdot V > G^{-1}(t)\right) = P\left(V > F^{-1}(t) - U, V > \frac{G^{-1}(t)}{U}\right)$$
$$= \mu\left(A(t)\right) \ge \mu\left(C(t)\right) = \frac{\left(1-G^{-1}(t)\right)^2}{2}$$

which, by the substitution $s = G^{-1}(t)$ or $t = G(s)$, gives

$$\lambda_U = \lim_{t\uparrow 1} \frac{1}{1-t} P\left(U+V > F^{-1}(t), U\cdot V > G^{-1}(t)\right) \ge \lim_{t\uparrow 1} \frac{\left(1-G^{-1}(t)\right)^2}{2(1-t)} = \lim_{s\uparrow 1} \frac{(1-s)^2}{2\left(1-G(s)\right)} = 1,$$

hence $\lambda_U = 1$, as stated. Note that by a Taylor expansion around the point $s = 1$, we have $G(s) = 1 - \frac{1}{2}(s-1)^2 + \mathcal{O}\left((s-1)^3\right)$, hence $\frac{2\left(1 - G(s)\right)}{(1-s)^2} = 1 + \mathcal{O}\left((s-1)\right)$.

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L = 0$.



Here we use the inequality

$$\mu\left(A(t)\right) \le \mu\left(A(t) \cup B(t)\right) = \frac{G^{-1}(t)}{F^{-1}(t)} \cdot F^{-1}(t) + \int_{\frac{G^{-1}(t)}{F^{-1}(t)}}^{F^{-1}(t)} \frac{G^{-1}(t)}{u}\, du = G^{-1}(t) + G^{-1}(t) \ln\left(\frac{F^{-1}(t)^2}{G^{-1}(t)}\right), 0 < t < 1$$

where $\mu\left(A(t)\right) = P\left(U + V \le F^{-1}(t),\, U \cdot V \le G^{-1}(t)\right)$.

Now, since $F^{-1}(t) \approx \sqrt{2t}$ for $t \downarrow 0$ and $\lim_{s \downarrow 0} G(s) = 0$, with the substitution $t = G(s)$ or $s = G^{-1}(t)$,

$$\lambda_L = \lim_{t \downarrow 0} \frac{\mu\left(A(t)\right)}{t} \le \lim_{t \downarrow 0} \frac{1}{t}\left[G^{-1}(t) + G^{-1}(t) \ln\left(\frac{F^{-1}(t)^2}{G^{-1}(t)}\right)\right] = \lim_{s \downarrow 0} \frac{s}{G(s)}\left[1 + \ln\left(\frac{F^{-1}(G(s))^2}{s}\right)\right]$$

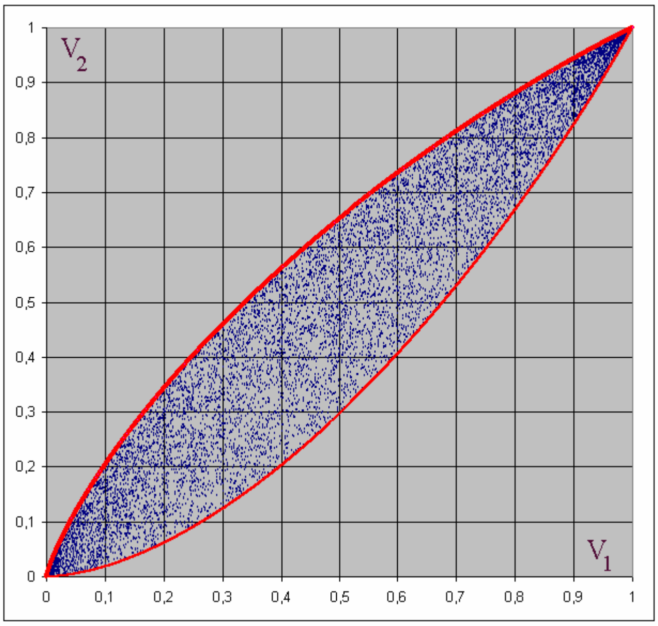
$$= \lim_{s \downarrow 0} \frac{s}{G(s)}\left[1 + \ln\left(\frac{2G(s)}{s}\right)\right] = \lim_{s \downarrow 0} \frac{1 + \ln(2)}{1 - \ln(s)} + \lim_{s \downarrow 0} \frac{\ln\left(1 - \ln(s)\right)}{1 - \ln(s)} = \lim_{z \uparrow \infty} \frac{\ln(1+z)}{1+z} = 0$$

by the substitution $s = e^{-z}$, i.e. $\lambda_L = 0$, as stated.

Case 2. In [12], this refers to Case 6. We consider $W_1(\mathbf{U}) = \min(U_1, U_2)$, $W_2(\mathbf{U}) = U_1 \cdot U_2$. It is easy to see that the c.d.f. G is given by

$$G(x) = (1 - \ln(x)) \cdot x, \ 0 < x \le 1 \quad \text{as above and}$$
$$F(x) = 1 - (1 - x)^2, \ 0 \le x \le 1.$$

The following graph shows 10.000 simulations of **V**.



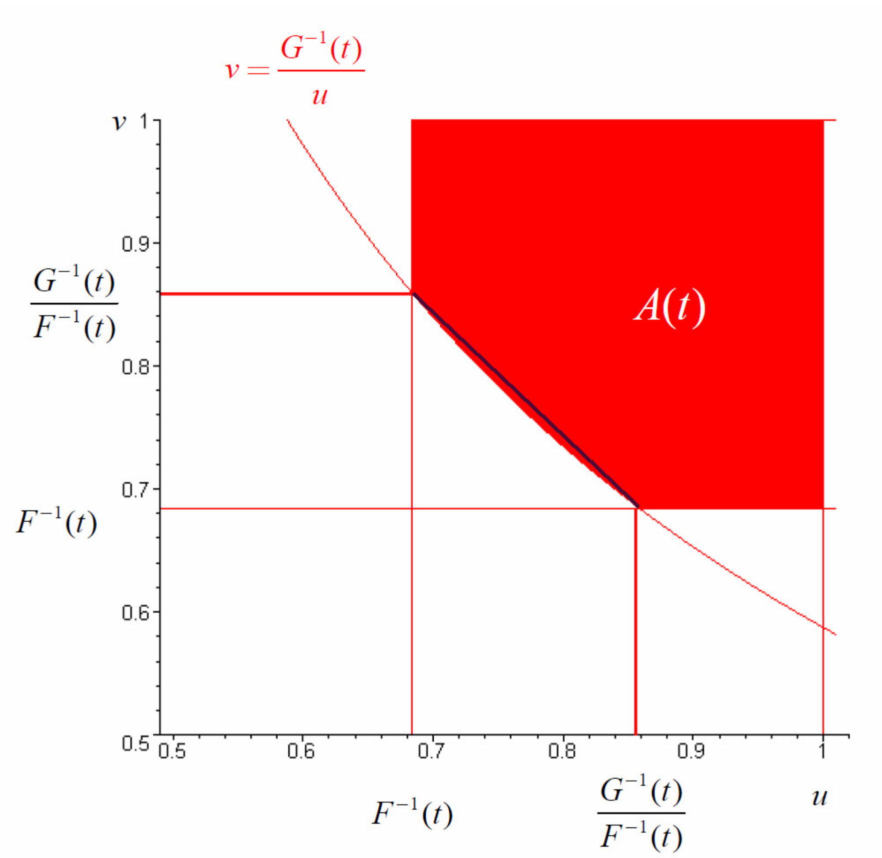
The red lines (u, v) represent the lower and upper envelopes of the copula, which are given by

$$v_{lower} = \left(1 - \sqrt{1-u}\right)^2 \cdot \left(1 - 2\ln\left(1 - \sqrt{1-u}\right)\right) \quad \text{and} \quad v_{upper} = \left(1 - \sqrt{1-u}\right) \cdot \left(1 - \ln\left(1 - \sqrt{1-u}\right)\right),$$

$0 < u < 1$, see [8].

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 2 \cdot \left(\sqrt{2} - 1\right)$.

We start again with some preliminary inequalitites.



We have

$$P\left(\min(U,V) > F^{-1}(t),\, U \cdot V \ge G^{-1}(t)\right) = P\left(U > F^{-1}(t),\, V > F^{-1}(t), V > \frac{G^{-1}(t)}{U}\right)$$

$$= \mu\left(A(t)\right) \ge \left(1 - F^{-1}(t)\right)^2 - \frac{\left(\frac{G^{-1}(t)}{F^{-1}(t)} - F^{-1}(t)\right)^2}{2} = 1 - t - \frac{\left(\frac{G^{-1}(t)}{F^{-1}(t)} - F^{-1}(t)\right)^2}{2}$$

or

$$\frac{P\left(\min(U,V) > F^{-1}(t), U \cdot V > G^{-1}(t)\right)}{1-t} \ge 1 - \frac{\left(\frac{G^{-1}(t)}{F^{-1}(t)} - F^{-1}(t)\right)^2}{2(1-t)},$$

hence, by the substitution $s = G^{-1}(t)$ or $t = G(s)$ as above,

$$\lambda_U = \lim_{t \to 1} \frac{P\left(\min(U,V) > F^{-1}(t), U \cdot V > G^{-1}(t)\right)}{1-t} \ge = 1 - \lim_{s \to 1} \frac{\left(\frac{s}{F^{-1}(G(s))} - F^{-1}(G(s))\right)^2}{2(1-G(s))}$$

$$= 1 - \lim_{s \to 1} \frac{K(s)}{2(1-G(s))} = 2 \cdot \left(\sqrt{2} - 1\right) = 0.828427125\ldots$$

with $K(s) = \left(\frac{s}{F^{-1}(G(s))} - F^{-1}(G(s))\right)^2$, $s = 0 \ldots 1.$

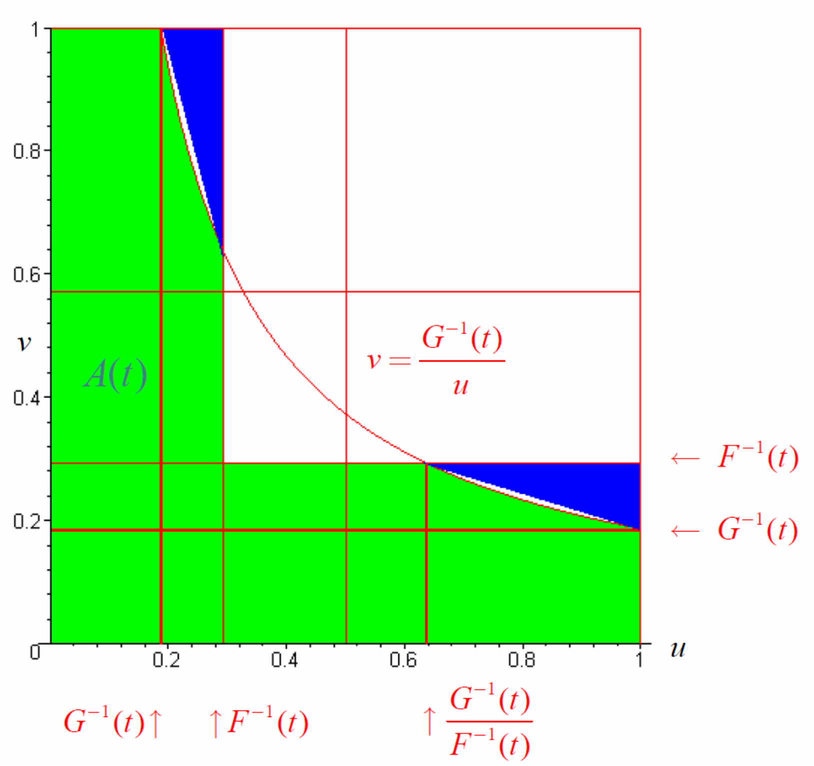
Note that by a Taylor expansion around the point $s=1$, we obtain $K(s)=\left(\sqrt{2}-1\right)^2(s-1)^2+\mathcal{O}\left((s-1)^3\right)$ and $2(1-G(s))=(s-1)^2+\mathcal{O}\left((s-1)^3\right)$, such that $1-\lim_{s\to 1}\frac{K(s)}{2(1-G(s))}=1-\left(\sqrt{2}-1\right)^2=2\cdot\left(\sqrt{2}-1\right).$

On the other side, using the fact that the graph of $v=\frac{G^{-1}(t)}{u}$ is convex and thus the map $v=2\sqrt{G^{-1}(t)}-u$ is a lower tangent at the point $u=\sqrt{G^{-1}(t)}$, we get an upper bound for the tail-dependence coefficient given by

$$\lambda_U\le 1-\lim_{s\to 1}\frac{\left(2\sqrt{G^{-1}(t)}-2F^{-1}(t)\right)^2}{2(1-t)}=1-\lim_{s\to 1}\frac{\left(2\sqrt{s}-2F^{-1}(G(s))\right)^2}{2(1-G(s))}$$
$$=1-\lim_{s\to 1}\frac{J(s)}{2(1-G(s))}=2\cdot\left(\sqrt{2}-1\right)$$

with $J(s)=\left(2\sqrt{s}-2F^{-1}(G(s))\right)^2$, which gives an identical estimate as above. Note that by a Taylor expansion around the point $s=1$, we obtain $J(s)=\left(\sqrt{2}-1\right)^2(s-1)^2+\mathcal{O}\left((s-1)^3\right)$, hence $\lambda_U=2\cdot\left(\sqrt{2}-1\right)=0.828427125\ldots$

The subsequent graph explains our arguments for the calculation of the coefficient λ_L of lower tail dependence, which is given by $\lambda_L=0$.
We start again with some preliminary inequalitites.



We have

$$\mu\left(A(t)\right) \le 1-\left(1-F^{-1}(t)\right)^2 - 2\left(F^{-1}(t)-G^{-1}(t)\right)\left(1-\frac{G^{-1}(t)}{F^{-1}(t)}\right) = t - 2\frac{\left(F^{-1}(t)-G^{-1}(t)\right)^2}{F^{-1}(t)},$$

hence

$$\lambda_L = \lim_{t\downarrow 0}\frac{P\left(\min(U,V)\le F^{-1}(t),\, U\cdot V \le G^{-1}(t)\right)}{1-t} = \lim_{t\downarrow 0}\frac{\mu\left(A(t)\right)}{t} \le 1 - 2\lim_{t\downarrow 0}\frac{\left(F^{-1}(t)-G^{-1}(t)\right)^2}{t\cdot F^{-1}(t)}$$
$$= 1 - 2\lim_{s\downarrow 0}\frac{\left(F^{-1}(G(s))-s\right)^2}{G(s)\cdot F^{-1}(G(s))} = 0.$$

Note that by a Taylor expansion of $F^{-1}(t)$ around the point $t=0$, we get $F^{-1}(t)=\frac{t}{2}+\mathcal{O}\left(t^3\right)$, hence $\left(F^{-1}(G(s))-s\right)^2 \approx \left(\frac{G(s)}{2}-s\right)^2$ and thus

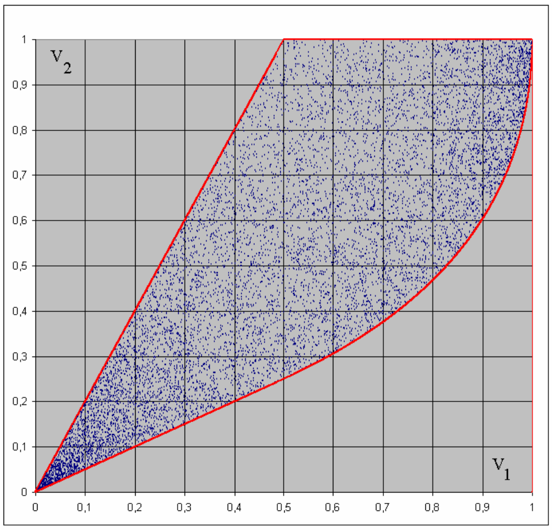
$$2\frac{\left(F^{-1}(G(s))-s\right)^2}{G(s)\cdot F^{-1}(G(s))} = \frac{\left(F^{-1}(G(s))-s\right)^2}{\frac{G(s)}{2}\cdot F^{-1}(G(s))} \approx \frac{\left(\frac{G(s)}{2}-s\right)^2}{\left(\frac{G(s)}{2}\right)^2} = \left(1-\frac{2s}{G(s)}\right)^2 = \left(1-\frac{2}{1-\ln(s)}\right)^2$$

with $\lim_{s\downarrow 0}\left(1-\frac{2}{1-\ln(s)}\right)^2 = 0$. This implies $\lambda_L = 0$, as stated.

Case 3. Here we consider a new case $W_1(\mathbf{U}) = U_1 + U_2$, $W_2(\mathbf{U}) = \max\left(U_1, U_2\right)$. It is easy to see (cf. Case 1) that the corresponding c.d.f.'s are given by

$$F(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x \le 1 \\ 1-2\left(1-\frac{x}{2}\right)^2, & 1 \le x \le 2. \end{cases} \quad \text{and}$$
$$G(x) = x^2,\ 0 \le x \le 1.$$

The following graph shows 10.000 simulations of **V**.



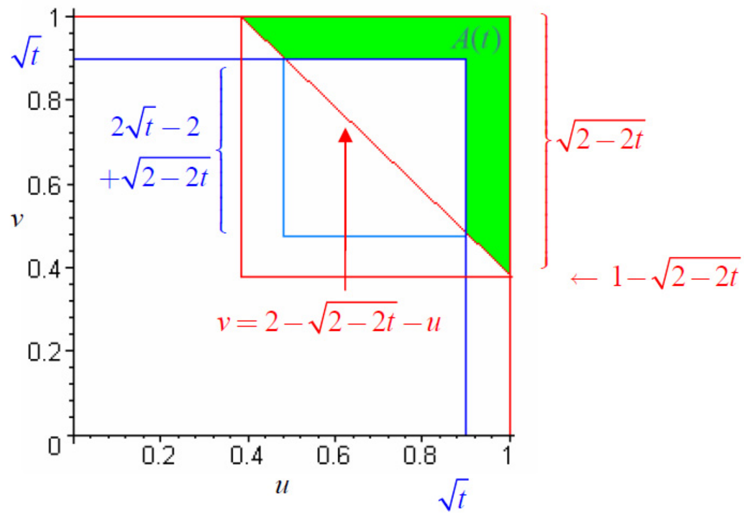
The red lines (u,v) represent the sharp lower and upper envelopes of the copula, which are given by

$$v_{lower} = \begin{cases} \dfrac{u}{2}, & \text{if } u \le \dfrac{1}{2} \\ \left(1-\sqrt{\dfrac{1-u}{2}}\right)^2, & \text{otherwise} \end{cases}$$

and $v_{upper} = \begin{cases} 2u, \text{ if } u \le \dfrac{1}{2} \\ 1, \ \text{ if } u > \dfrac{1}{2} \end{cases}$,

The lower bound is reached if V_1 and V_2 are close to each other, while the upper bound is reached if one of V_1 or V_2 is close to zero.
The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 0$.

We start again with some preliminary inequalitites.



We have, for $t > \dfrac{1}{2}$,

$$P\left(\max(U,V) > G^{-1}(t),\, U+V > F^{-1}(t)\right) = P\left(\max(U,V) > \sqrt{t},\, V > 2-\sqrt{2-t}-U\right)$$

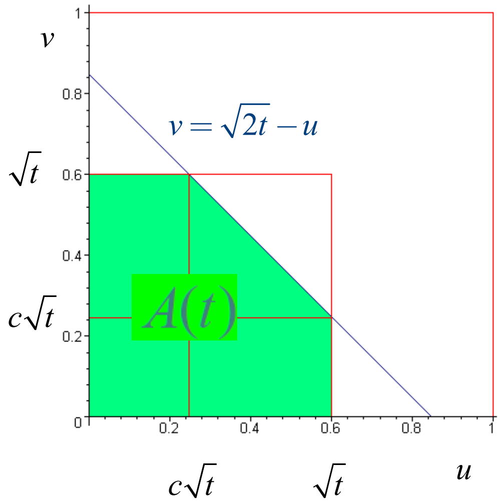
$$= \mu\left(A(t)\right) = \frac{1}{2}\left(\sqrt{2-2t}^{\,2} - \left(2\sqrt{t}-2+\sqrt{2-2t}\right)^2\right)$$

or, by a Taylor expansion around the point $t=1$,

$$\mu\left(A(t)\right) = \sqrt{2}(1-t)^{3/2} + \mathcal{O}\left((1-t)^2\right), \text{ hence } \lambda_U = \lim_{t\to 1}\frac{\mu\left(A(t)\right)}{1-t} = 0, \text{ as stated.}$$

The subsequent graph explains our arguments for the calculation of the coefficient λ_U of upper tail dependence, which is given by $\lambda_U = 2\left(\sqrt{2}-1\right) = 0{,}828427\ldots$.

We start again with some preliminary inequalitites.



We have, for $t > \frac{1}{2}$, with $c = \sqrt{2}-1$,

$$P\left(\max(U,V) \le G^{-1}(t),\, U+V \le F^{-1}(t)\right) = P\left(\max(U,V) \le \sqrt{t},\, V \le \sqrt{2t}-U\right)$$

$$= \mu\left(A(t)\right) = t - \frac{\left\{(1-c)\sqrt{t}\right\}^2}{2} = t\left(1-\frac{(1-c)^2}{2}\right) = 2ct$$

and hence $\lambda_L = \lim_{t\downarrow 0}\frac{\mu\left(A(t)\right)}{t} = 2c = 2\left(\sqrt{2}-1\right) = 0{,}828427\ldots$, as stated.

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