Fundamental Journal of Mathematics and Mathematical Sciences
p-ISSN: 2395-7573; e-ISSN: 2395-7581
Volume 19, Issue 1, 2025, Pages 89-95
This paper is available online at http://www.frdint.com/
Published online January 30, 2025

A SIMPLE RECURSIVE REPRESENTATION OF THE FAULHABER SERIES

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Abstract

We present a simple elementary recursive representation of the so called Faulhaber series $\sum_{k=1}^{n} k^N$ for integer n and N, without reference to Bernoulli numbers or polynomials.

1. Introduction

A well-known historical problem is the explicit evaluation of the so called Faulhaber series $\sum_{k=1}^{n} k^N$ for integer n and N, see, e.g., Knuth [2] who also coined the wording Faulhaber series or formula. In the modern literature on this topic, an explicit representation of this expression is

Keywords and phrases: history of mathematics, Faulhaber series.

2020 Mathematics Subject Classification: 01A45, 11B37, 11B65, 11B68, 11B83.

Received January 17, 2025; Accepted January 22, 2025

given on the basis of Bernoulli numbers and Bernoulli polynomials, see [1], [2] and [3]. In this note, we present a simple recursive representation of the Faulhaber series without reference to Bernoulli numbers or polynomials.

2. Main Result

Denote $s(n, N) := \sum_{k=1}^{n} k^N$ for integer n and N. Then there holds

$$s(n, N) := \sum_{k=0}^{n} k^N = \frac{(n+1)^{N+1} - \sum_{j=0}^{N-1} \binom{N+1}{j} s(n, j)}{N+1}. \tag{1}$$

Proof.

$$(n+1)^{N+1} = \sum_{k=0}^{n} \left[\sum_{j=0}^{N+1} \binom{N+1}{j} k^j - k^{N+1} \right] = \sum_{k=0}^{n} \sum_{j=0}^{N} \binom{N+1}{j} k^j$$

$$= \sum_{k=0}^{n} (N+1) k^N + \sum_{k=0}^{n} \sum_{j=0}^{N-1} \binom{N+1}{j} k^j$$

$$= (N+1) \sum_{k=0}^{n} k^N + \sum_{j=0}^{N-1} \binom{N+1}{j} \sum_{k=0}^{n} k^j$$

$$= (N+1) \sum_{k=0}^{n} k^N + \sum_{j=0}^{N-1} \binom{N+1}{j} s(n, j), \tag{2}$$

which by rearrangement leads to

$$s(n, N) = \sum_{k=0}^{n} k^n = \frac{(n+1)^{N+1} - \sum_{j=0}^{N-1} \binom{N+1}{j} s(n, j)}{N+1}. \tag{3}$$

Obviously,

$$s(n,\ N) = \sum_{k=0}^{n} k^N = \frac{(n+1)^{N+1} - \binom{N+1}{N-1} s(n,\ N-1) - \sum_{j=0}^{N-2} \binom{N+1}{j} s(n,\ j)}{N+1} \tag{4}$$

which implies that $s(n,\ N)$ is a polynomial in n of degree $N+1$ with leading term $\frac{(n+1)^{N+1}}{N+1}$ or more precisely, $s(n.\ N) = \frac{n^{N+1}}{N+1} + \frac{n^N}{2} + \mathcal{O}(n^{N-1})$, cf. [3], p. 3.

An evaluation of [1] for consecutive values of N leads to:

$$s(n,\ 0) = \sum_{k=0}^{n} 1 = n+1,$$

$$s(n,\ 1) = \frac{(n+1)^2 - s(n,\ 0)}{2} = \frac{n(n+1)}{2},$$

$$s(n,\ 2) = \frac{(n+1)^3 - s(n,\ 0) - 3s(n,\ 1)}{3}$$

$$= \frac{(n+1)^3 - (n+1) - 3\,\frac{n(n+1)}{2}}{3} = \frac{n(n+1)(2n+1)}{6},$$

$$s(n,\ 3) = \frac{n^2(n+1)^2}{4}, \tag{5}$$

$$s(n,\ 4) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30},$$

$$s(n,\ 5) = \frac{n^2(n+1)^2(2n^2+2n-1)}{12},$$

$$s(n,\ 6)=\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42},$$

$$s(n,\ 7)=\frac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{24},$$

$$s(n,\ 8)=\frac{n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)}{90},$$

$$s(n,\ 9)=\frac{n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)}{20},$$

$$s(n,\ 10)=\frac{\left[\begin{array}{l}n(n+1)(2n+1)(n^2+n-1)\\ \times(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)\end{array}\right]}{66},$$

$$s(n,\ 11)=\frac{\left[\begin{array}{l}n^2(n+1)^2(2n^8+8n^7+4n^6-16n^5\\ \qquad -5n^4+26n^3-3n^2-20n+10)\end{array}\right]}{24},$$

$$s(n,\ 12)=\frac{\left[\begin{array}{l}n(n+1)(2n+1)(105n^{10}+525n^9+525n^8-1050n^7\\ \qquad -1190n^6+2310n^5\\ \qquad +1420n^4-3285n^3\\ \qquad -287n^2+2073n-691)\end{array}\right]}{2730},$$

$$s(n,\ 13)=\frac{\left[\begin{array}{l}n^2(n+1)^2(30n^{10}+150n^9+125n^8-400n^7\\ \qquad -326n^6+1052n^5+367n^4\\ \qquad -1786n^3+202n^2+1382n-691)\end{array}\right]}{420},$$

$$s(n, 14) = \frac{\left[\begin{array}{l} n(n+1)(2n+1)(3n^{12} + 18n^{11} + 24n^{10} \\ \qquad - 45n^9 - 81n^8 + 144n^7 + 182n^6 \\ \qquad - 345n^5 - 217n^4 + 498n^3 \\ \qquad + 44n^2 - 315n + 105) \end{array}\right]}{90},$$

$$s(n, 15) = \frac{\left[\begin{array}{l} n^2(n+1)^2(3n^{12} + 18n^{11} + 21n^{10} \\ \qquad - 60n^9 - 83n^8 + 226n^7 \\ \qquad + 203n^6 - 632n^5 - 226n^4 \\ \qquad + 1084n^3 - 122n^2 - 840n + 420) \end{array}\right]}{48}, \tag{6}$$

$$s(n, 16) = \frac{\left[\begin{array}{l} n(n+1)(2n+1)(15n^{14} + 105n^{13} + 175n^{12} \\ \qquad - 315n^{11} - 805n^{10} + 1365n^9 \\ \qquad + 2775n^8 - 4845n^7 - 6275n^6 \\ \qquad + 11835n^5 - 17145n^3 - 1519n^2 \\ \qquad + 7485n^4 + 10851n - 3617) \end{array}\right]}{510},$$

$$s(n, 17) = \frac{\left[\begin{array}{l} n^2(n+1)^2(10n^{14} + 70n^{13} + 105n^{12} \\ \qquad - 280n^{11} - 565n^{10} + 1410n^9 \\ \qquad + 2165n^8 - 5740n^7 - 5271n^6 \\ \qquad + 16282n^5 + 5857n^4 - 27996n^3 \\ \qquad + 3147n^2 + 21702n - 10851) \end{array}\right]}{180},$$

$$s(n,\,18)=\frac{\left[\begin{array}{l} n(n+1)(2n+1)(105n^{16}+840n^{15}+1680n^{14} \\ \qquad -2940n^{13}-9996n^{12}+16464n^{11} \\ \qquad +48132n^{10}-80430n^{9}-167958n^{8} \\ \qquad +292152n^{7}+380576n^{6}-716940n^{5} \\ \qquad -454036n^{4}+1039524n^{3}+92162n^{2} \\ \qquad -658005n+219335) \end{array}\right]}{3990},$$

$$s(n,\,19)=\frac{\left[\begin{array}{l} n^2(n+1)^2(42n^{16}+336n^{15}+616n^{14}-1568n^{13} \\ \qquad -4263n^{12}+10094n^{11}+22835n^{10} \\ \qquad -55764n^{9}-87665n^{8}+231094n^{7} \\ \qquad +213337n^{6}-657768n^{5}-236959n^{4} \\ \qquad +1131686n^{3}-127173n^{2} \\ \qquad -877340n+438670) \end{array}\right]}{840},$$

$$s(n,\,20)=\frac{\left[\begin{array}{l} n(n+1)(2n+1)(165n^{18}+1485n^{17}+3465n^{16} \\ \qquad -5940n^{15}-25740n^{14}+41580n^{13} \\ \qquad +163680n^{12}-266310n^{11}-801570n^{10} \\ \qquad +1335510n^{9}+2806470n^{8}-4877460n^{7} \\ \qquad -6362660n^{6}+11982720n^{5}+7591150n^{4} \\ \qquad -17378085n^{3}-1540967n^{2} \\ \qquad +11000493n-3666831) \end{array}\right]}{6930}.$$

The final calculations have been supported by the computer algebra system MAPLE. Seemingly, $s(n,\,N)$ can be represented as $s(n,\,N)=n^2(n+1)^2P(n,\,N-3)$ for uneven $N\geq 3$ and $s(n,\,N)=n(n+1)(2n+1)P(n,\,N-2)$ for even $N\geq 2$ where $P(n,\,K)$ is a polynomial of degree K. This can be shown empirically at least for $N = 2, \ldots, 100$.

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