Reflections on a canonical construction principle for multivariate copula models

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Abstract We consider a canonical construction principle for multivariate copula models on the basis of independent standard random variables which is in particular well suited for Monte Carlo Studies.

1. Introduction. There are many approaches to copula modelling in the literature, cf. e.g. the papers listed in the References below. Now for our investigations, let $\mathbf{U}=\{U_k\}_{k\in\mathbb{N}}$ be a sequence of independent standard random variables, i.e. each U_k has a continuous uniform distribution over the interval $[0,1]$. Let further $T_1,\cdots,T_n$, $\mathrm{n}\in\mathbb{N}$ be real continuous functions over $\mathbb{R}^{\mathbb{N}}$ and $V_i=T_i(\mathbf{U})$ for $i=1,\cdots,n$ with a continuous uniform distribution over $[0,1]$ each. Then $\mathbf{V}=(V_1,\cdots,V_n)$ is a representative of an *n*-dimensional copula.

Note that if $W_i=T_i(\mathbf{U})$ is not directly uniformly distributed then $V_i=F_i(W_i)$ is so if F_i denotes the c.d.f. of W_i.

2. Particular Cases. Consider the following special cases of a construction as indicated in the Introduction.

Case 1. Let $n=2$ and $T_1(\mathbf{U})=U_1$, $W_2=T_2(\mathbf{U})=\alpha U_1+(1-\alpha)U_2$, $0<\alpha\leq\frac{1}{2}$. it can easily be shown that the c.d.f. F_2 is given by

$$F_2(x,\alpha)=\begin{cases}\dfrac{x^2}{2\alpha(1-\alpha)} & 0\leq x\leq\alpha\\ \dfrac{x}{1-\alpha}-\dfrac{\alpha}{2(1-\alpha)} & \alpha\leq x\leq 1-\alpha\\ 1-\dfrac{(1-x)^2}{2\alpha(1-\alpha)} & 1-\alpha\leq x\leq 1\end{cases},\qquad (1)$$

The following graphs show 5.000 simulations of **V** each, for various values of α.

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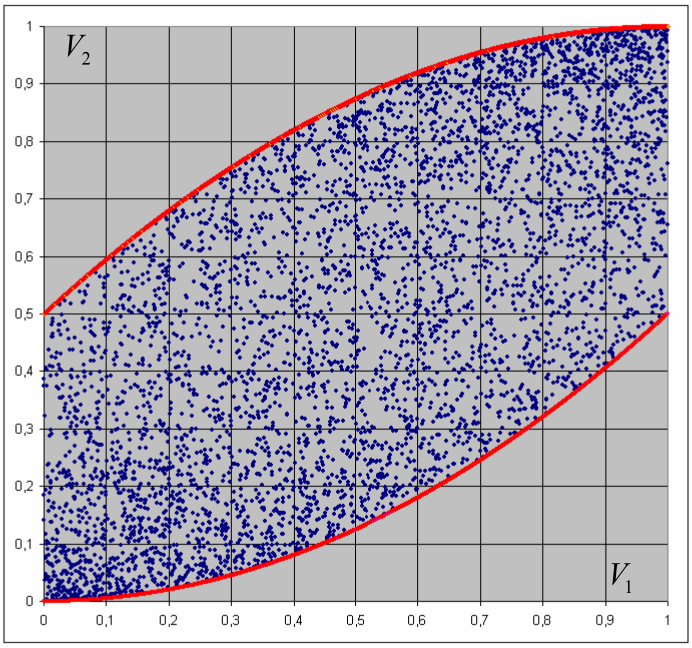
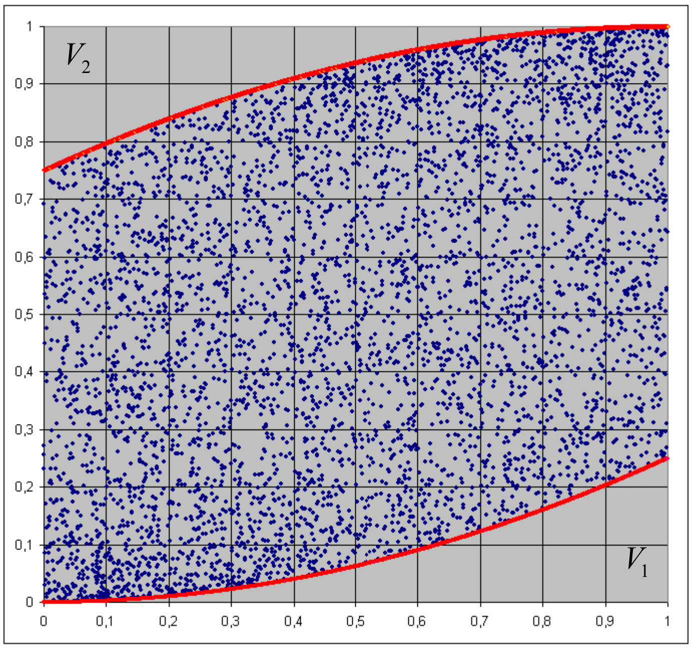
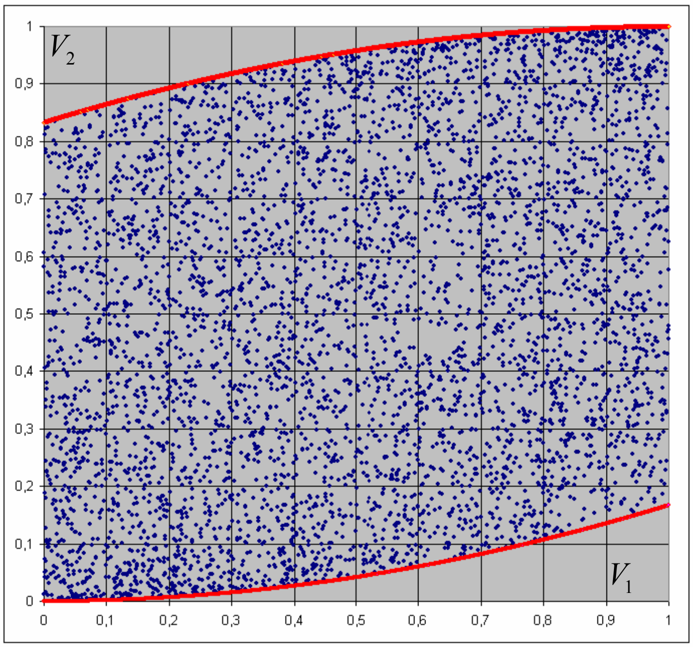


Scatterplot of $\mathbf{V}, \alpha = \frac{1}{2}$



Scatterplot of $\mathbf{V}, \alpha = \frac{1}{3}$



Scatterplot of $\mathbf{V}, \alpha = \frac{1}{4}$

The red lines (u,v) represent the lower and upper envelopes of the copula, which are given by

$v_{lower} = \beta u^2$ and $v_{upper} = 1-\beta(1-u)^2$, $0<u<1$ with $\beta = \frac{\alpha}{2(1-\alpha)}$. This follows from (1) since $W_2 \geq \alpha U_1 \in [0,\alpha]$ and $V_2 = F_2(W_2,\alpha) \geq \frac{W_2^2}{2\alpha(1-\alpha)} \geq \frac{\alpha U_1^2}{2(1-\alpha)} = \beta U_1^2$ which implies the lower envelope. Note also that if U_2 is close to zero, then V_2 is close to $\frac{\alpha U_1^2}{2(1-\alpha)} = \beta U_1^2$ which implies that the lower envelope is sharp. The upper envelope follows by symmetry reasons.

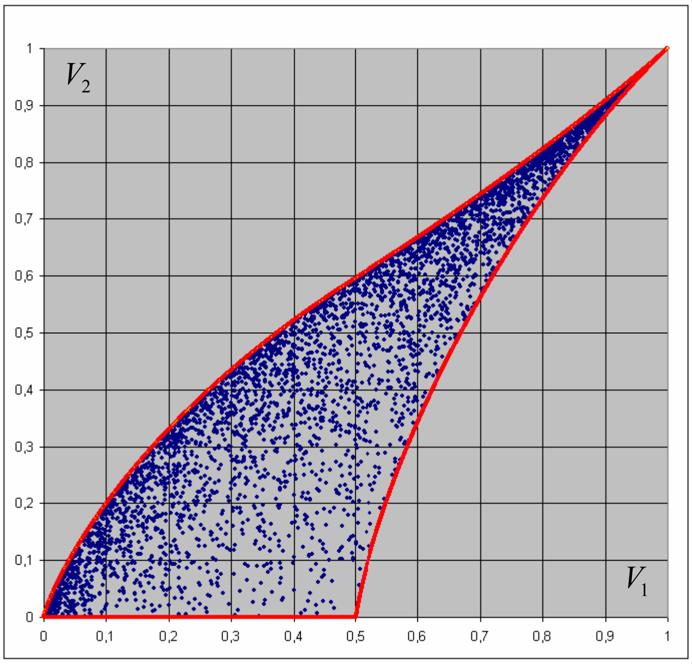
Case 2. Let $n=2$ and $W_1(\mathbf{U}) = U_1 + U_2$, $W_2(\mathbf{U}) = U_1 \cdot U_2$. It is easy to see that the c.d.f. F_2 is given by

$$F_2(x) = (1-\ln(x))\cdot x, \quad 0 < x \leq 1 \quad \text{and}$$

$$F_1(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x \leq 1 \\ 1-2\left(1-\frac{x}{2}\right)^2, & 1 \leq x \leq 2 \end{cases} \tag{2}$$

(cf. Case 1 for $\alpha = \frac{1}{2}$).

This follows from the observation that $-\ln(W_2(\mathbf{U}))$ represents the sum of two independent standard exponentially distributed random variables, hence is gamma-distributed. The following graph shows 5.000 simulations of $\mathbf{V}$.



Scatterplot of **V**

The red lines (u,v) represent the lower and upper envelopes of the copula, which are given by

$$v_{lower} = \begin{cases} 0, & \text{if } u \leq \frac{1}{2} \\ \left(1-\sqrt{2-2u}\right)\left(1+\ln\left(1-\sqrt{2-2u}\right)\right), & \text{otherwise} \end{cases}$$

and $$v_{upper} = \begin{cases} \frac{u}{2}\cdot\left(1-\ln\left(\frac{u}{2}\right)\right) & \text{if } u \leq \frac{1}{2} \\ \left(1-\frac{1}{2}\sqrt{1-u}\right)\cdot\left(1-2\ln\left(1-\frac{1}{2}\sqrt{1-u}\right)\right), & \text{if } u > \frac{1}{2} \end{cases},$$

$0 < u < 1$. Note also that if U_2 is close to U_1, then the upper envelope is reached. The lower envelope is reached if U_2 is close to 1.

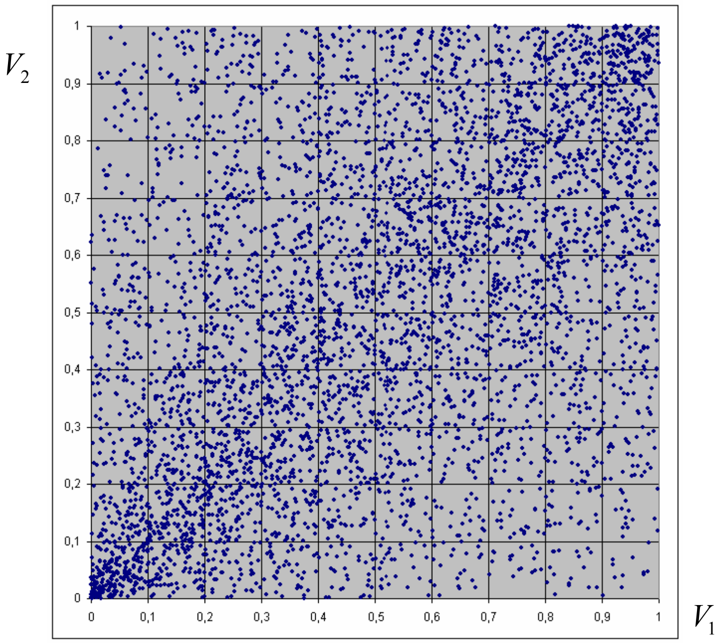
Case 3. Let $n = 3$ and $T_1(\mathbf{U}) = U_1$, $T_2(\mathbf{U}) = \left(U_1 \cdot U_2\right)^{U_3}$. Note that $V_2 = T_2(\mathbf{U})$ is already uniformly distributed over $[0,1]$ since for $0 < x < 1$

$$P\left(V_3 \leq x\right) = P\left(-\ln\left(V_3\right) \geq -\ln(x)\right) = P\left(-\ln\left(U_1\right) - \ln\left(U_2\right) \geq \frac{-\ln(x)}{U_3}\right)$$

$$= \int_0^1 \left(1 - \frac{\ln(x)}{w}\right) \cdot x^{1/w} dw = w \cdot x^{1/w} \Big|_{w=0}^{w=1} = x$$

(note that $-\ln\left(U_1\right) - \ln\left(U_2\right)$ is gamma-distributed).

The following graph shows 5.000 simulations of **V**.



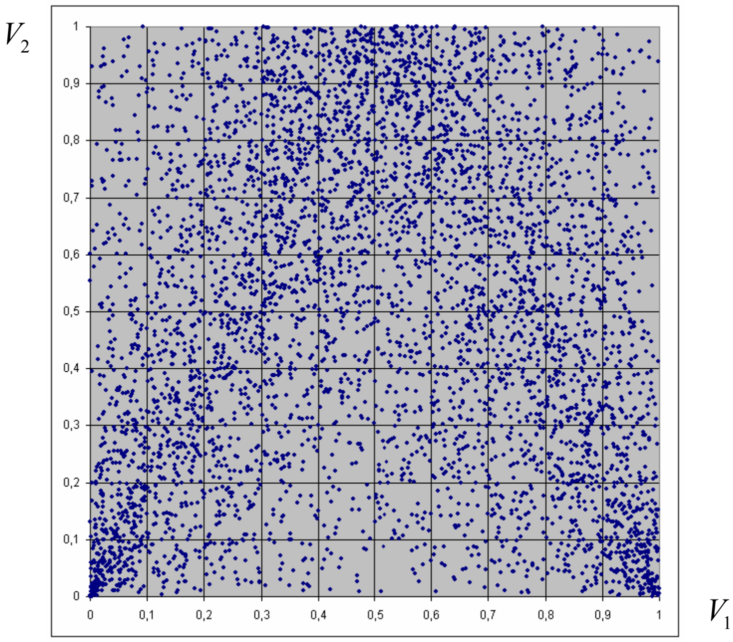
Scatterplot of **V**

Case 4. Let $n=3$ and $W_1(\mathbf{U})=\dfrac{U_1}{U_2}$, $T_2(\mathbf{U})=\left(U_1\cdot U_2\right)^{U_3}$. Note that the c.d.f. F_1 of $W_1(\mathbf{U})$ is given by

$$F_1(x)=\begin{cases}\dfrac{x}{2}, & x\le 1\\ 1-\dfrac{1}{2x}, & x\ge 1\end{cases} \tag{3}$$

while $T_2(\mathbf{U})$ is already continuous uniformly distributed over $[0,1]$, cf. Case 4.

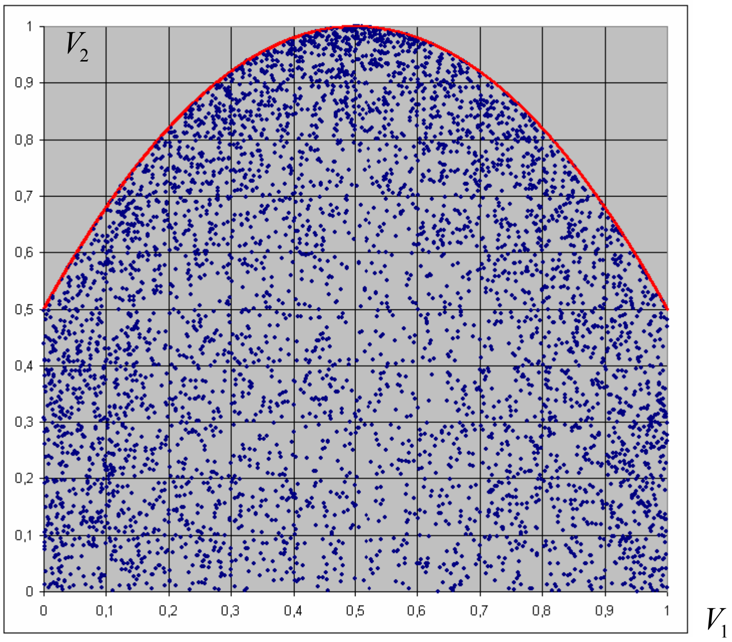
The following graph shows 5.000 simulations of **V**.



Scatterplot of **V**

Case 5. Let $n=2$ and $W_1(\mathbf{U})=\frac{U_1}{U_2}$, $W_2(\mathbf{U})=U_1+U_2$. For the corresponding c.d.f.s, see Cases 4 and 2.

The following graph shows 5.000 simulations of $\mathbf{V}$.

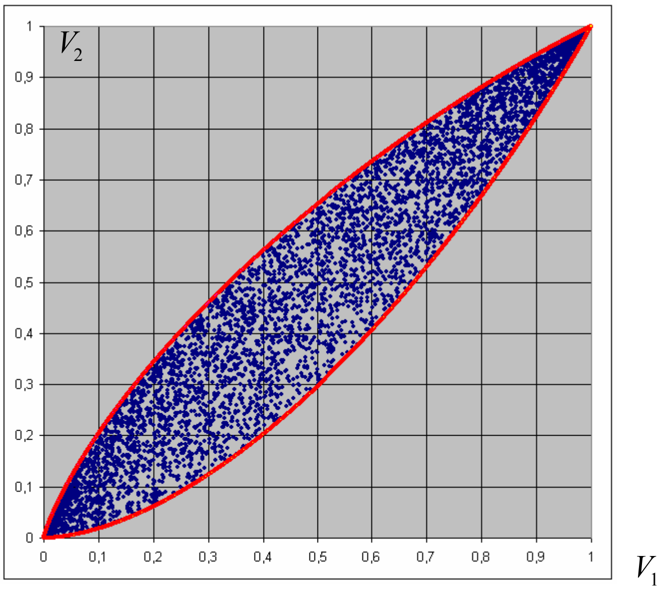


Scatterplot of $\mathbf{V}$

The red line (u,v) represents the upper envelope of the copula, which is given by $v=1-2\left(u-\frac{1}{2}\right)^2$. Note that if U_2 is close to 1 then V_1 is close to $\frac{U_1}{2}$ and V_2 is close to $1-2\left(1-\frac{U_1+U_2}{2}\right)^2 \approx 1-2\left(\frac{1}{2}-V_1\right)^2$ which implies that the envelope is sharp.

Case 6. Let $n=2$ and $W_1(\mathbf{U})=\min(U_1,U_2)$, $W_2(\mathbf{U})=U_1\cdot U_2$. Clearly (cf. (2)), $F_1(x)=1-(1-x)^2$ and $F_2(x)=(1-\ln(x))\cdot x,\ 0<x\leq 1$.

The following graph shows 5.000 simulations of $\mathbf{V}$.



Scatterplot of $\mathbf{V}$

The red lines (u,v) represent the lower and upper envelopes of the copula, which are given by

$$v_{lower} = \left(1-\sqrt{1-u}\right)^2 \cdot \left(1-2\ln\left(1-\sqrt{1-u}\right)\right) \qquad \text{and} \qquad v_{upper} = \left(1-\sqrt{1-u}\right)\cdot\left(1-\ln\left(1-\sqrt{1-u}\right)\right),$$

$0<u<1$. This can be seen as follows: we have $V_1 = 1-(1-U_1 \wedge U_2)^2$ with $a \wedge b = \min(a,b)$ for real a,b or $U_1 \wedge U_2 = 1-\sqrt{1-V_1}$. It follows

$$\left(1-\sqrt{1-V_1}\right)^2 = (U_1 \wedge U_2)^2 \le U_1 \cdot U_2 \le U_1 \wedge U_2 = 1-\sqrt{1-V_1}$$

and, since the map $z \to g(z) := z\cdot(1-\ln(z))$ with $g'(z) = -\ln(z) > 0,\ z \in (0,1]$ is monotonically increasing we have

$$\left(1-\sqrt{1-V_1}\right)^2 \cdot \left(1-\ln\left(\left(1-\sqrt{1-V_1}\right)^2\right)\right) \le V_2 = g\left(U_1 \cdot U_2\right) \le \left(1-\sqrt{1-V_1}\right)\cdot\left(1-\ln\left(1-\sqrt{1-V_1}\right)\right)$$

which proves the statement. Note that if U_1 is close to U_2 then the lower envelope becomes sharp while the upper envelope becomes sharp if U_1 or U_2 is close to zero.

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