Outline of the course:

I Basics on geometry: spaces (manifolds with corners)
functions (polyhomogeneous)
blow-up

II Basics on analysis: · Mellin transform
· Integration (push-forward)

IV... Applications to PDE problems
- b-calculus
- heat kernel
...

Literature: · R. Melrose (MIT): online book (unfinished)
· DG: Basics of the b-calculus

I. Basics on geometry

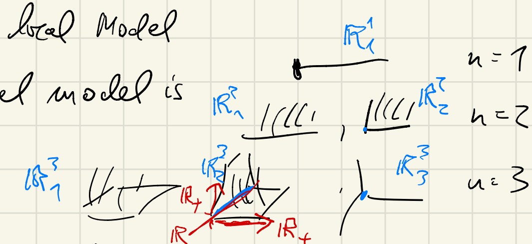
I.1 Manifolds with corners, smooth maps

Why manifolds? · Natural setting for coordinate free understanding of things
· even if we start in $\mathbb{R}^n$, we'll get outside via blow-ups.

Why corners? We want to have the possibility that some coordinates are ≥ 0.

Recall manifold:
space locally like $\underline{\mathbb{R}^n}$ (or $\Omega \subset \mathbb{R}^n$ open)
local Model

M.w. corners: local model is



Def: $\mathbb{R}^n_k := \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$ $\quad (0 \leq k \leq n)$

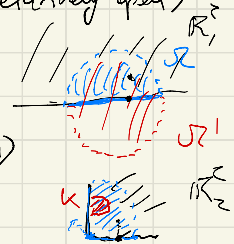
$\mathbb{R}_+ := [0, \infty)$

Need: What does smoothness mean for maps on $\mathbb{R}^n_k$?

Open sets: $\Omega \subset \mathbb{R}^n_k$ is open (or relatively open)

$\forall p \in \Omega \ \exists \varepsilon > 0$

$\{q \in \mathbb{R}^n_k : |p-q| < \varepsilon\} \subset \Omega$



$\Omega \subset \mathbb{R}^n_k$ open $\iff \exists\, \Omega' \subset \mathbb{R}^n$ open (in $\mathbb{R}^n$)

$\Omega = \Omega' \cap \mathbb{R}^n_k$

Ω' = an extension of Ω.

Def: Let $\Omega \subset \mathbb{R}^n_k$ be open, $u : \Omega \to \mathbb{C}$.

u is smooth $:\iff u|_{\Omega^\circ}$ smooth (C^∞) and all $D^\alpha u$ extend continuously to Ω.

Notation: $u \in C^\infty(\Omega)$

Notation: $(\mathbb{R}^n_k)^\circ = (0,\infty)^k \times \mathbb{R}^{n-k}$, $\quad \Omega^\circ := \Omega \cap (\mathbb{R}^n_k)^\circ$

$D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j} = \frac{1}{i}\partial_{x_j}$, $\quad D^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, $\quad \alpha = (\alpha_1 \dots \alpha_n) \in \mathbb{N}_0^n$.

∂^α similar

Note: If $u \in C^\infty(\Omega)$ then $D^\alpha u$ is bounded on all $K \subset\subset \Omega$ $\ \forall \alpha$

(compact subset)

Conversely, given $u^\circ \in C^\infty(\Omega^\circ)$

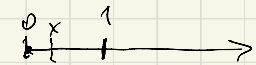
such that $D^\alpha u^\circ$ is bounded on $K \cap \Omega^\circ$

for all $K \subset\subset \Omega$.

Then u° has a unique extension to $u \in C^\infty(\Omega)$.

[Therefore, we usually don't distinguish u and u°].

Reason: Say $n = k = 1$; $\Omega = \mathbb{R}_+$



$u^\circ : (0,\infty) \to \mathbb{C}$, all deriv. bdd.

$$u^\circ(x) = u^\circ(1) + \int_1^x \partial_x u^\circ(t)\,dt \qquad \forall x > 0.$$

$\Rightarrow \lim_{x \to 0} u^\circ(x) =: u(0)$ exists.

Similar for $(u^\circ)^{(m)}$.

Rem: Let Ω' be an extension of Ω, $u : \Omega \to \mathbb{C}$.

Then $u \in C^\infty(\Omega) \iff \exists\, u' \in C^\infty(\Omega')$ so that $u'|_\Omega = u$.

$\boxed{u \text{ smooth} \iff u \text{ has a smooth extension}}$ (Seeley's extension theorem)

Def: a) $\Omega \subset \mathbb{R}^n_k$ open. $F : \Omega \to \mathbb{R}^N$ is smooth $\Leftrightarrow$ $F = (F_1, \dots, F_N)$, each $F_i : \Omega \to \mathbb{R}$ is smooth.

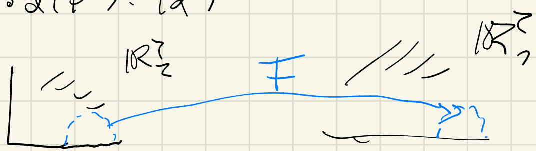
b) $\Omega_1 \subset \mathbb{R}^n_{k_1}$, $\Omega_2 \subset \mathbb{R}^n_{k_2}$.

$F : \Omega_1 \to \Omega_2$ is a diffeomorphism $:\Leftrightarrow$ F is bijective and F, F^{-1} are smooth.

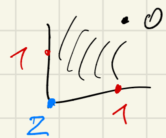
Note: If F is a diffeo then for each $p \in \Omega$, $dF|_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.

(since $dF \circ d(F^{-1}) = id$)

Ex:



Def: If $p \in \mathbb{R}^n_k$ then codim p = number of boundary hyperplanes containing p.



Notation: Coordinates on $\mathbb{R}^n_k = \mathbb{R}^k_+ \times \mathbb{R}^{n-k}$

$(x_1, \dots, x_k)$ boundary coord. $\quad (y_1, \dots, y_{n-k})$ interior coord.

$x_i \geq 0$

$y_i \in \mathbb{R}$.

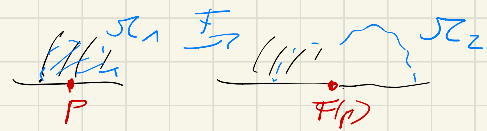
Then if $p = (x_1, \dots, x_k, y_1, \dots, y_{n-k})$

$\text{codim}\, p = \#\{ i : x_i = 0 \}$

Lemma: Diffeomorphisms preserve codimension:

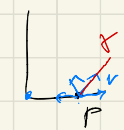
$F : \Omega_1 \to \Omega_2$ diffeo, $p \in \Omega_1 \Rightarrow$

$\text{codim}\, p = \text{codim}\, F(p)$.

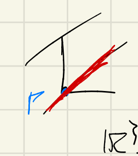


Pf: For $\Omega \subset \mathbb{R}^n_k$, $p \in \Omega$ let

$T_p^+\Omega = \{ v \in \mathbb{R}^n : \exists \varepsilon > 0$
$\exists \gamma : [0, \varepsilon] \to \Omega$ smooth curve,
$\dot\gamma(0) = v \}$



Then $\text{codim}\, p$ = codimension of the largest vector subspace contained in $T_p^+\Omega$.



Clearly, $F : \Omega_1 \to \Omega_2 \Rightarrow dF_p : T_p^+\Omega_1 \to T_{F(p)}^+\Omega_2$, $\quad \mathbb{R}^n \to \mathbb{R}^n$.

Since dF_p is an isomorphism it follows that $\text{codim}\, p = \text{codim}\, F(p)$. q.e.d.

Def:

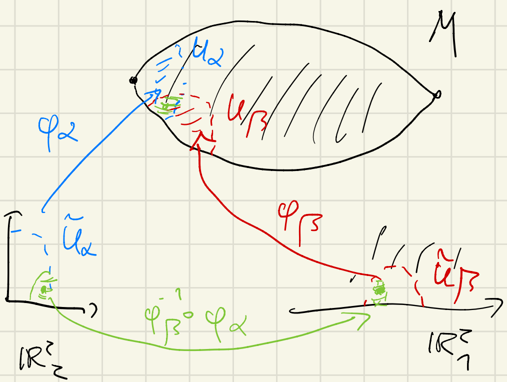
a) A t-manifold (of dimension n) is a topological space X which is Hausdorff, second countable and so that:
there is an open cover $(U_\alpha)_{\alpha \in A}$ of X and for each α a homeomorphism

$$\varphi_\alpha : \tilde{U}_\alpha \to U_\alpha \quad , \quad \tilde{U}_\alpha \subset \mathbb{R}^n_k \text{ open for some } k$$

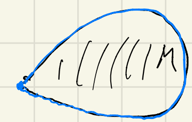
so that all the coord. change maps

$$\varphi_\beta^{-1} \circ \varphi_\alpha$$

are diffeomorphisms.

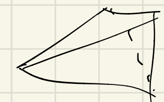


b) A manifold with corners is a t-manifold all of whose boundary hypersurfaces are submanifolds.

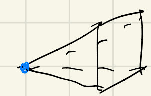


t-manifold
not a man. w. c.

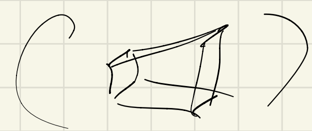
Ex: $\mathbb{R}^n_k$ is a mwc.

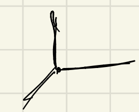


is a mwc $(n=3)$.



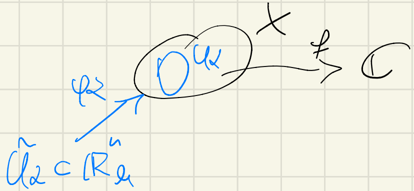
not a t-mfd.



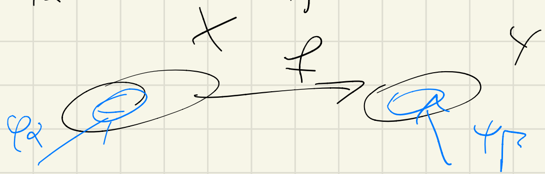


Rem: Other people use different definitions
(ex: t-mfd = mwc)

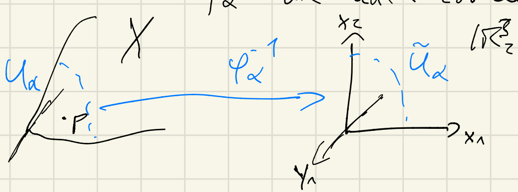
Def: X t-manifold, $u : X \to \mathbb{C}$.
u smooth $:\Leftrightarrow f \circ \varphi_\alpha$ is smooth $\forall \alpha$.



Def: X, Y t-mflds, $F : X \to Y$ smooth if
$\psi_\beta^{-1} \circ f \circ \varphi_\alpha$ is smooth $\forall \alpha, \beta$.



Words: · φ_α are called local charts for X
· $(\varphi_\alpha)_{\alpha \in A}$ is called atlas for X.
· φ_α^{-1} are called coordinate systems for X.



$\varphi_\alpha^{-1}(p) = (x_1(p), x_2(p), y_1(p))$.

· A coord. system is centred at p if it maps $p \mapsto 0$.

Def: Let X be a t-manifold.
a) For $p \in X$ define codim p as codim $\varphi_\alpha^{-1}(p)$ for any chart where $p \in U_\alpha$.
b) A face of codimension k of X is the closure of a connected component of $\{p \in X : \operatorname{codim} p = k\}$.
c) A boundary hypersurface of X is a face of codim. 1.

Ex: