§3. Spaces of functions and distributions

<u>Definition 3.1</u> A locally convex vector space V is a topological vector space with topology induced by a family $(p_i)_{i\in I}$ of semi-norms, ie V is equipped with the coarsest topology for which all mappings

$$p_{i,y}: \quad V \ni x \longmapsto p_i(x-y) \in \mathbb{K} \quad , i\in I, y \in V$$

are continuous. A topological base of open nbds is given in this topology by $\mathcal{U}_{i_1,\dots,i_r,\varepsilon}(y) = \{ x\in V \mid \forall j=1,\dots,r: p_{i_j}(x-y) < \varepsilon \}$.

<u>Important examples</u>:

0) Banach- and Hilbert-spaces are trivially locally convex
1) Frechet-spaces: ie those locally convex v.s. with
   - countably many semi-norms
   - complete vectorspace.

3) LF-spaces: (E_j) sequence of Frechet spaces, $E_1 \subset E_2 \subset E_3 \cdots$; $E := \bigcup\limits_n E_n$ with inductively defined topology such that $E_n \hookrightarrow E$ are continuous. Topology consists of all $\mathcal{U} \subset E$ st $\mathcal{U} \cap E_n$ open for all n.

$E_i \subset E_{i+1}$ continuously

We want to discuss the following spaces from the point of view of locally convex vector spaces:

$\longrightarrow$ $\mathcal{S}(\mathbb{R}^n) = \{ f\in C^\infty(\mathbb{R}^n) \mid \forall \alpha,\beta \in \mathbb{Z}^n_+ : p_{\alpha\beta}(f) < \infty \}$
where $p_{\alpha\beta}(f) = \sup\limits_x | x^\alpha D^\beta f(x)|$

$\longrightarrow$ For $\Omega \subset \mathbb{R}^n$ offen $\mathcal{E}(\Omega) := C^\infty(\Omega)$

$\longrightarrow$ For $\Omega \subset \mathbb{R}^n$ open $\mathcal{D}(\Omega) := \{ f\in C^\infty(\Omega) \mid \operatorname{supp} f \text{ cpt} \}$
the so-called "<u>test functions</u>"

Frechet - structure on $\mathcal{S}(\mathbb{R}^n)$

(ii) $(p_{\alpha\beta})_{\alpha,\beta\in\mathbb{Z}_+^n}$ are countably many semi-norms that turn $\mathcal{S}(\mathbb{R}^n)$ into a Frechet-space.

Frechet - structure on $\mathcal{E}(\Omega)$

Choose a countable exhaustion of Ω by compact sets $(K_j)_{j\in\mathbb{N}}$ ie:

$$K_1 \subset \mathring{K}_2 \subset K_2 \subset \mathring{K}_3 \subset \cdots$$

$$\Omega = \bigcup_{n=1}^{\infty} K_n$$

only diff-n

Then $(p_{\alpha,K_n})_{\alpha\in\mathbb{Z}^n_+,\, n\in\mathbb{N}}$ defines Frechet-topology on $\mathcal{E}(\Omega)$.

LF ~~Frechet~~ - structure on $\mathcal{D}(\Omega)$

This is somewhat harder. Put for $K\subset\Omega$ compact

$$\mathcal{D}_K(\Omega) = \{ f\in C_0^\infty(\Omega) \mid supp(f)\subset K\}$$

ie $\mathcal{D}_{K'}$ induces on $\mathcal{D}_K$ the original topology

$\mathcal{D}_K(\Omega)$ with semi-norms $(p_{\alpha,K})_{\alpha\in\mathbb{Z}^n_+}$ is a Frechet space.
If $K\subset K'$ then $\mathcal{D}_K(\Omega)\hookrightarrow\mathcal{D}_{K'}(\Omega)$ is a top inclusion.
Choose an exhaustion of Ω by compact sets as for $\mathcal{E}(\Omega)$. Then

$$\mathcal{D}(\Omega) = \bigcup_{n=1}^{\infty}\mathcal{D}_{K_n}(\Omega)$$

is an LF-space.

Definition 3.1 A semi-norm $p: \mathcal{D}(\Omega)\to\mathbb{R}^+$ is said to be cts if for each cpt $K_n\subset\Omega$ there exist constants C, N s.t. for $f\in\mathcal{D}_{K_n}(\Omega)$

$$p(f) \le C\cdot\sum_{|\alpha|\le N} p_{\alpha,K_n}(f) = C\cdot\sum_{|\alpha|\le N}\sup_{x\in K_n}|D^\alpha f(x)|$$

(continuity means for a norm bdedness by a finite family of seminorms)

Proposition 3.2

1) Topology of $\mathcal{D}(\Omega)$ is independent of the choice of an exhaustion (K_n).
2) $\mathcal{U} \subset \mathcal{D}(\Omega)$ is open iff for each cpt $K \subset \Omega$, $\mathcal{U} \cap \mathcal{D}_K(\Omega)$ open.
3) A seminorm p on $\mathcal{D}(\Omega)$ is cts iff $p \restriction \mathcal{D}_K(\Omega)$ is cts for each cpt $K \subset \Omega$, in other words $\mathcal{D}_K(\Omega) \hookrightarrow \mathcal{D}(\Omega)$ is a "topol. inclusion"
4) A sequence $(f_n) \subset \mathcal{D}(\Omega)$ converges to $f \in \mathcal{D}(\Omega)$ iff there exists cpt $K \subset \Omega$ such that

$$\begin{aligned} &\cdot\ \forall n:\ \operatorname{supp} f_n \subset K \\ &\cdot\ \forall \alpha \in \mathbb{Z}_+^n:\ D^\alpha f_n \xrightarrow{n\to\infty} D^\alpha f \text{ uniformly.} \end{aligned} \qquad (*)$$

Proof: (3) If $p \restriction \mathcal{D}_K(\Omega)$ is cts for all cpt $K \subset \Omega$, then in particular for all K_n. Hence p is a cts semi-norm on $\mathcal{D}(\Omega)$ by definition. Conversely, let p be a cts seminorm on $\mathcal{D}(\Omega)$. Let $K \subset \Omega$ be cpt. Pick K_N from the exhaustion of Ω st.

$$K \subset \mathring{K}_N \subset K_N.$$

Then by definition $p \restriction \mathcal{D}_{K_N}(\Omega)$ is cts and hence

$$\Rightarrow p \leq C \cdot \max_{j=1}^{r} p_{\alpha_j, K_N} \quad \text{for some } \alpha_1, \dots, \alpha_r.$$

$$\Rightarrow p \restriction \mathcal{D}_K(\Omega) \leq C \cdot \max_{j=1}^{r} p_{\alpha_j, K}$$

and hence $p \restriction \mathcal{D}_K(\Omega)$ is continuous. ✓

(4) Consider $(f_n) \subset \mathcal{D}(\Omega)$ satisfying $(*)$. Then clearly $\operatorname{supp} f \subset K$ $\{(*)$ in particular asserts $\|f_n - f\|_\infty \to 0\}$ and hence $\forall N:\ p_{N,K}(f_n - f) \xrightarrow{n\to\infty} 0$.
By Def 3.1 $p(f - f_n) \to 0$ for any cts semi-norm on $\mathcal{D}(\Omega) \Rightarrow f_n \to f$ in $\mathcal{D}(\Omega)$.

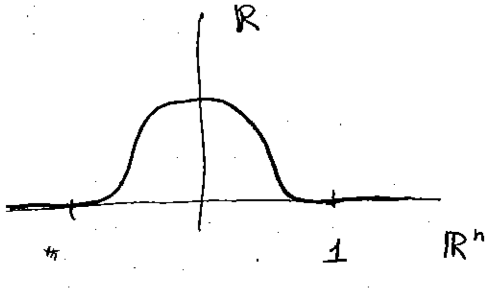
We omit proofs of the other statements.

<u>Other examples of locally convex vector spaces</u>

- $C_0^k(\Omega) = \bigcup_{n=0}^{\infty} C^k_{K_n}(\Omega)$ is an LF-space
- $C^k(\Omega)$ is a Frechet-space with seminorms $(p_{\alpha K})_{\alpha \in \mathbb{Z}^n_+}$ such that $|\alpha| \leq k$, $K \subset \Omega$ compact.
- $L^p_{comp}(\Omega) = \{ f \in L^p(\Omega) \,/\, f \text{ has a cptly supported representative} \}$
$= \bigcup_{n=0}^{\infty} L^p(K_n)$ is an LF-Space of Banach spaces
- $L^p_{loc}(\Omega) = \{ f : \Omega \to \mathbb{C} \,/\, f \text{ measurable and for } K \subset \Omega \text{ cpt } \int_K |f|^p < \infty \}$
is a Frechet-space with seminorms $\|f\|_{p, K_n} := \left(\int_{K_n} |f|^p \right)^{1/p}$.

§4. Convolution and regularization

Fix $\rho \in C_0^\infty(\mathbb{R}^n)$ such that



- $0 \leq \rho$
- $\int \rho = 1$
- $\rho(x) = 0$ if $\|x\| > 1$

For any $\lambda > 0$ define

$$\rho_\lambda(x) := \lambda^{-n} \rho\left(\frac{x}{\lambda}\right)$$

Certainly: $\int \rho_\lambda = 1$; but $\operatorname{supp} \rho_\lambda \subset B_\lambda(0)$ is ~~largely~~ changed.
This section studies important applications of convolution by ρ_λ.

Corollary $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ dense

Proof: $f_n \longrightarrow f \in \mathcal{E}(\Omega)$ means explicitly that on each $K \subset \Omega$ compact and for each $\alpha \in \mathbb{Z}_+^n$

$$p_{\alpha,K}(f_n - f) = \sup_{x \in K} |D^\alpha f_n(x) - D^\alpha f(x)| \underset{n \to \infty}{\longrightarrow} 0 .$$

Now let $f \in \mathcal{E}(\Omega)$, Consider exhaustion of Ω by (K_n)

$$K_1 \subset \mathring{K}_2 \subset K_2 \subset \mathring{K}_3 \subset \cdots$$

and pick $(\varphi_n) \subset C_0^\infty(\Omega)$ st:
- $\varphi_n \restriction K_n \equiv 1$
- $\varphi_n \restriction K_{n+1}^c \equiv 0$

Then for each fixed cpt $K \subset \Omega$ there exists n_0 st $K \subset K_n$ for $n \geq n_0$ and hence $p_{\alpha,K}(f \cdot \varphi_n - f) = p_{\alpha,K}(f - f) = 0$
Since $(f \cdot \varphi_n) \subset \mathcal{D}(\Omega)$, statement follows. ▨

(there is no need for $\rho_\lambda * f$ convolution)

Theorem 4.1

Let X be one of the following spaces:

$$L^p(\mathbb{R}^n),\ L^p_{loc}(\mathbb{R}^n),\ L^p_{comp}(\mathbb{R}^n);\ 1\le p<\infty$$
$$C_0^K(\mathbb{R}^n),\ C^K(\mathbb{R}^n),\ \mathcal{D}(\mathbb{R}^n),\ \mathcal{E}(\mathbb{R}^n),\ \mathcal{S}(\mathbb{R}^n).$$

Then for each $f\in X$, $\rho_\lambda * f$ converges in X to f as $\lambda\to 0$.

Proof: a) $X=C^K(\mathbb{R}^n)$ with the natural Frechet-topology.

Let $K\subset\mathbb{R}^n$ be compact and $K_1:=\{x\in\mathbb{R}^n \mid \operatorname{dist}(x,K)\le 1\}$ cpt.

For $0<\lambda<1$ and $x\in K$ we find: $(|\alpha|<K)$

$$|D^\alpha[\rho_\lambda * f](x)-D^\alpha f(x)|$$
$$=|\rho_\lambda * D^\alpha f(x)-D^\alpha f(x)|$$
$$=\left|\int \rho_\lambda(y)\{D^\alpha f(x-y)-D^\alpha f(x)\}\,dy\right|$$
$$\le \sup_{\|y\|\le\lambda}|D^\alpha f(x-y)-D^\alpha f(x)|$$

ie $\rho_\lambda * f\to f$ uniformly on compact subsets (natural Frechet top of $C^K(\mathbb{R}^n)$)

$$\Rightarrow \|D^\alpha(\rho_\lambda * f)-D^\alpha f\|_{\infty,K}\le \sup_{\substack{x,x'\in K_1\\ \|x-x'\|\le\lambda<1}}|D^\alpha f(x)-D^\alpha f(x')|$$

This can be made $<\varepsilon$ for any $\varepsilon>0$ for λ sufficiently small, since $D^\alpha f$ is continuous for $|\alpha|\le K$.

b) $X=\mathcal{E}(\mathbb{R}^n)=C^\infty(\mathbb{R}^n)$ follows by a similar argument.

c) $X=L^p(\mathbb{R}^n)\ni f$, $\varepsilon>0$. Choose $g\in C_0(\mathbb{R}^n)$ st $\|f-g\|_{L^p}<\varepsilon/3$

Then we use the general theorem on convolution:

$$u\in\mathcal{L}^1(\mathbb{R}^n),\ v\in\mathcal{L}^p(\mathbb{R}^n)\Rightarrow \|u*v\|_{L^p}\le\|u\|_{L^1}\cdot\|v\|_{L^p}$$

$$\Rightarrow \|\rho_\lambda * f - f\|_p \leq \|\rho_\lambda * (f-g)\|_p + \|\rho_\lambda * g - g\|_p + \|g-f\|_p$$

$$\leq \underbrace{\|\rho_\lambda\|_1}_{=1} \cdot \underbrace{\|f-g\|_p}_{<\varepsilon/3} + \underbrace{\|f-g\|_p}_{<\varepsilon/3} + \|\rho_\lambda * g - g\|_p$$

$$\leq \frac{2}{3}\varepsilon + \|\rho_\lambda * g - g\|_p$$

By (a) $\rho_\lambda * g \xrightarrow{\lambda \to 0} g$ uniformly $\Rightarrow \|\rho_\lambda * g - g\|_p \xrightarrow{\lambda \to 0} 0$

since $\operatorname{supp} \rho_\lambda * g \subset (\operatorname{supp} g)_\lambda = \{x \in \mathbb{R}^n \mid \operatorname{dist}(x, \operatorname{supp} g) < \lambda\}$.

Other cases are left to the reader. $\square$

~~Corollary 4.2 $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$, $\mathcal{D}(\Omega) \subset \mathcal{E}(\Omega)$ dense (direct consequence of a) with $(\rho_\lambda * f)_\lambda$ approx. sequence)~~

<u>Corollary 4.3</u> Let $f \in L^1_{loc}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open.

If $\int_\Omega f\varphi = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, then $f = 0$

<u>Proof</u>: It suffices to show $f \cdot \psi = 0$ for all $\psi \in \mathcal{D}(\Omega)$.
Hence consider $\tilde{f} = f \cdot \psi$, extended by zero to L^1-function/class in $\mathbb{R}^n$. For $\varphi \in \mathcal{D}(\mathbb{R}^n)$ we have $\psi\varphi \in \mathcal{D}(\Omega)$. Hence

$$\int_{\mathbb{R}^n} \tilde{f}\varphi = \int_\Omega f(\psi\varphi) = 0$$

$$\Rightarrow \rho_\lambda * \tilde{f}(x) = \int_{\mathbb{R}^n} \tilde{f}(y)\, \rho_\lambda(x-y)\, dy = 0$$

$$\Rightarrow \text{By Theorem 4.1 } \tilde{f} = \lim_{\lambda \to 0} \rho_\lambda * \tilde{f} = 0 \text{ in } L^1(\mathbb{R}^n)$$

$\square$

Overview over the lecture

Fourier transform

$\mathcal{F} : L^2 \to L^2$ isometry
$\mathcal{F} : \mathcal{S} \to \mathcal{S}$
interchanges D^α, ξ^α.

Symbols

$D \in \mathrm{Diff}^m(M)$
$Df = \mathcal{F}^{-1}[\sigma_D(x,\xi)\hat{f}]$
for $f \in \mathcal{S}(\mathbb{R}^n)$
σ_D inv-ble $\Rightarrow$ D elliptic

Sobolev-Spaces

- Sobolev-embedding
- Rellich-embedding
- Trace-theorem

$H^s(\mathbb{R}^n)$, for $s \in \mathbb{N}$ order understood as L^2-regularity under differentiation

Distributions

$\mathcal{S}', \mathcal{E}', \mathcal{D}'$

Oszillatory integrals

Pseudo-differential operators $\rightarrow$

- PDE
- heat eqn
- wave eqn
- Hodge decomp (elliptic regularity)

etc.

§ 3. Distributions

$\Omega \subseteq \mathbb{R}^n$ open, connected

Definition 3.1

- space of "distributions" $\mathcal{D}'(\Omega) := \{ T : \mathcal{D}(\Omega) \to \mathbb{C} \mid \text{linear, cts} \}$
- space of "distr. with cpt. support" $\mathcal{E}'(\Omega) := \{ T : \mathcal{E}(\Omega) \to \mathbb{C} \mid \text{lin, cts} \}$
- space of "tempered distributions" $\mathcal{S}'(\mathbb{R}^n) := (\mathcal{S}(\mathbb{R}^n))'$.

Remark 3.2

a) $T : \mathcal{D}(\Omega) \to \mathbb{C}$ is continuous in the sense that for any $(f_n) \subset \mathcal{D}(\Omega)$, $f_n \to f \Rightarrow \lim_{n\to\infty} T(f_n) = T(f)$.
Similarly for the other distribution spaces.

b) $\mathcal{D}'(\Omega)$, $\mathcal{E}'(\Omega)$, $\mathcal{S}'(\mathbb{R}^n)$ are equipped with the weak topology of dual spaces, ie: $(T_n) \subset \mathcal{D}'(\Omega)$ converges to T iff $\forall \varphi \in \mathcal{D}(\Omega)$: $\lim_{n\to\infty} T_n(\varphi) = T(\varphi)$.

Satz 3.3 c) $\mathcal{D}'(\Omega)$ may equivalently be characterized as follows:
$T : \mathcal{D}(\Omega) \to \mathbb{C}$ is a distribution iff T is linear and $\forall K \subset \Omega$ cpt $\exists N \in \mathbb{Z}_+, C \in \mathbb{R}_+ : |T(f)| \leq C \cdot \sum_{|\alpha| \leq N} p_{\alpha K}(f)$ for all $f \in \mathcal{D}_K(\Omega)$.

Satz 3.3 d) $\mathcal{E}'(\Omega)$ may equivalently be characterized as follows:
$T : \mathcal{E}(\Omega) \to \mathbb{C}$ is a distribution with cpt supp iff T linear and $\exists K \subset \Omega$ cpt, $N \in \mathbb{Z}_+$, $C \in \mathbb{R}_+$ $\forall f \in \mathcal{E}(\Omega)$:

$$|T(f)| \leq C \cdot \sum_{|\alpha| \leq N} \underbrace{p_{\alpha K}(f)}_{(\text{recall}) = \|\partial^\alpha f\|_{\infty, K}}$$

e) The notion of "distr. with cpt support" will become clear once we define support of a distribution

Example 3.4

Each $f \in L^1_{loc}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'(\Omega)$ by

$$T_f(\varphi) := \int_\Omega f\varphi \quad (\text{note: } \operatorname{supp}\varphi \subset K \text{ cpt})$$

In fact, the inclusion $L^1_{loc}(\Omega) \hookrightarrow \mathcal{D}'(\Omega);\ f \longmapsto T_f$ is a continuous injection:

- $|T_f(\varphi)| = |\int_\Omega f\varphi| \leq (\int_K |f|) \cdot \sup_{x\in K}|\varphi(x)|$

  and hence for any $f_n \to f \in L^1_{loc}(\Omega)$; $T_{f_n} \to T_f$ in the weak topology of the dual space.

- injectivity follows from corollary in the previous section.

Example 3.5 "The δ-Distribution"

$$\delta_a : \mathcal{D}(\Omega) \longrightarrow \mathbb{C}\ ;\ \varphi \longmapsto \varphi(a)$$

a) there exists no $f \in L^1_{loc}(\Omega)$ such that $T_f = \delta_a$.

b) However δ_a may be approximated by (T_{f_n}), $(f_n) \subset L^1_{loc}(\Omega)$:

Proof: a) Assume $\delta_a = T_f$ and then for all $\varphi \in \mathcal{D}(\Omega\setminus\{a\})$

$$0 = \delta_a(\varphi) = \int_{\Omega\setminus\{a\}} f\varphi \Rightarrow f = 0 \text{ almost everywhere on } \Omega\setminus\{a\}$$

by corollary from the previous section. Hence $f = 0$ a.e. on Ω and then $\int_\Omega f\varphi = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ $\lightning$.

b) Set $T_{\rho_\lambda}(\varphi) = \int_\Omega \rho_\lambda(y)\cdot\varphi(y)\,dy$

$$= \int_{\mathbb{R}^n} \rho_\lambda(y)\cdot\varphi(y)\,dy \quad \text{for } \lambda \text{ suff. small} > 0$$

$$= (\rho_\lambda * \varphi)(0) \xrightarrow[\lambda\to 0]{} \varphi(0) = \delta_0(\varphi)$$

This proves $\lim_{\lambda \to 0} T_{\rho_\lambda} = \delta_0$ in $\mathcal{D}'(\Omega)$ $\blacksquare$

<u>Proposition 3.6</u> (without proof) $\mathcal{D}(\Omega) \hookrightarrow \mathcal{D}'(\Omega)$ dense, ie for any $T \in \mathcal{D}'(\Omega)$ there exists $(f_n) \subset \mathcal{D}(\Omega)$ such that

$$T_{f_n} \xrightarrow{n \to \infty} T \quad \text{in } \mathcal{D}'(\Omega)\text{-topology}$$

$$\Downarrow \qquad \text{ie } \forall \varphi \in \mathcal{D}(\Omega): \lim_{n\to\infty} \int_\Omega f_n \cdot \varphi = T(\varphi)$$

Approximation of δ_0 by T_{ρ_λ} is just one particular example of this general fact.

Differentiation of distributions

<u>Definition 3.7</u> For $T \in \mathcal{D}'(\Omega)$ $\{$or $\mathcal{E}'(\Omega)$, $\mathcal{S}'(\mathbb{R}^n)\}$, $\alpha \in \mathbb{Z}_+^n$

define $D^\alpha T(\varphi) := (-1)^{|\alpha|} T(D^\alpha \varphi)$

for any $\varphi \in \mathcal{D}(\Omega)$.

<u>Remark 3.8</u> This definition is consistent with differentiation of functions under the embeddings

$$C^k(\Omega) \hookrightarrow L^1_{loc}(\Omega) \longrightarrow \mathcal{D}'(\Omega)$$

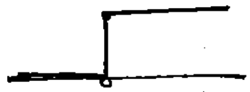
Namely, for $f \in C^k(\Omega)$, $\varphi \in \mathcal{D}(\Omega)$

$$\begin{aligned} D^\alpha T_f(\varphi) &= (-1)^{|\alpha|} T_f(D^\alpha \varphi) \\ &= (-1)^{|\alpha|} \int_\Omega f \cdot D^\alpha \varphi \\ &\underset{\text{(Int. by parts)}}{=} \int_\Omega D^\alpha f \cdot \varphi = T_{D^\alpha f}(\varphi) \end{aligned}$$

ie $D^\alpha T_f = T_{D^\alpha f}$ for $f \in C^k(\Omega)$.

Hence we can now diff. L^1_{loc}-functions in the sense of distributions

Examples 3.9

a) $H(x) = \mathbb{1}_{\mathbb{R}_+}(x)\,,\ x \in \mathbb{R}$  step function

$$\partial_x H(\varphi) = -\,H(\varphi') = -\int_0^\infty \varphi'$$

$$= \varphi(0) = \delta_0(\varphi)$$

$\Rightarrow \boxed{\partial_x H = \delta_0}$ derivative of a step function is the Delta-distribution.

b) $f(x) = \begin{cases} \log(x), & x > 0 \\ 0, & x < 0 \end{cases}\,,\quad x \in (-1,1)$

Clearly $f \in L^1_{loc}(-1,1)$. For $x \neq 0$:

$$f'(x) = \begin{cases} 1/x, & x > 0 \\ 0, & x < 0 \end{cases}\,,\quad f' \notin L^1_{loc}(-1,1)$$

Hence $T_{f'}$ is a priori not defined. We compute by defn:

$$\partial_x T_f(\varphi) = -\int_0^1 \log x \cdot \varphi'(x)$$

$$\equiv -\lim_{\varepsilon \to 0} \int_\varepsilon^1 \varphi'(x) \log x$$

$$= \lim_{\varepsilon \to 0} \left\{ \varphi(\varepsilon)\log(\varepsilon) + \int_\varepsilon^1 \frac{\varphi(x)}{x} \right\}$$

$$= \lim_{\varepsilon \to 0} \left\{ [\underbrace{\varphi(\varepsilon) - \varphi(0)}_{=\mathcal{O}(\varepsilon),\ \varepsilon \to 0}] \log \varepsilon + \int_\varepsilon^1 \frac{\overbrace{\varphi(x) - \varphi(0)}^{=\mathcal{O}(x),\ x \to 0}}{x} \right.$$

$$= \int_0^1 \frac{\varphi(x) - \varphi(0)}{x}\, dx$$

§ Further properties of distributions

<u>Lemma 4.12</u> Let $I \subset \mathbb{R}$ be an open interval and $T \in \mathcal{D}'(I)$ with $\partial T = 0$. Then $T = c$ (almost everywhere) is a constant function.

<u>Proof:</u> Fix $\psi \in \mathcal{D}(I)$ with $\int_I \psi = 1$ and put $(I = (a,b))$ for $\varphi \in \mathcal{D}(I)$:

$$\Phi(x) := \int_a^x \left[\varphi - \left(\int_I \varphi\right)\psi\right]$$

Clearly $\Phi \in C^\infty(I)$ with $\operatorname{supp}\Phi \subset$ convex hull of $\operatorname{supp}\varphi \cup \operatorname{supp}\psi$.
Hence $\Phi \in \mathcal{D}(I)$ (compact support) and:

$$\begin{aligned} 0 = \langle \underbrace{\partial T}_{=0}, \Phi \rangle &= -\langle T, \partial\Phi \rangle \\ &= -\langle T, \varphi - \left(\int_I \varphi\right)\psi \rangle \\ &= \langle T, \psi \rangle \int_I \varphi - \langle T, \varphi \rangle \end{aligned}$$

$$\Rightarrow \langle T, \varphi \rangle = \langle T, \psi \rangle \int_I \varphi = \int_I c\varphi$$ ☐

<u>Theorem 4.13</u> (without proof)

(a) Let $I \subset \mathbb{R}$ be an open interval and let $T \in \mathcal{D}'(I^n)$ with $\partial_n T = 0$. Then there exists $T_1 \in \mathcal{D}'(I^{n-1})$ s.t. for $\varphi \in \mathcal{D}(I^n)$:

$$\langle T, \varphi \rangle = \langle T_1, \int_I \varphi(\cdot, x_n)\, dx_n \rangle$$

(b) Let $\Omega \subset \mathbb{R}^n$ be open and $T \in \mathcal{D}'(\Omega)$ with $\partial_j T = 0$ for $j = 1, \dots, n$. Then T is a locally constant function.

Fourier transform of tempered distributions

<u>Definition 4.14</u> For $T \in \mathcal{S}'(\mathbb{R}^n)$ setzte / define $\mathcal{F}T \in \mathcal{S}'(\mathbb{R}^n)$

$$\langle \mathcal{F}T, \varphi \rangle := \langle T, \mathcal{F}\varphi \rangle \quad [\mathcal{F}T_f = T_{\mathcal{F}f}]$$

<u>FACT:</u> $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \circlearrowleft$ is a (linear) topological isom-m, ie $\mathcal{F}$ is cts and $\mathcal{F}^{-1}$ is cts as well in the weak top.

This fact is a general consequence of the following theorem:

Ü <u>Theorem 4.15</u> Let $A : X \to Y$ be a linear cts mapping between locally convex vector spaces. Then A defines a linear op $A' : Y' \to X'$ by

$$\langle A'y', x \rangle := \langle y', Ax \rangle \text{ for all } x \in X, y' \in Y'.$$

If A is a topological isom-m (ie A, A^{-1} both cts) then A' is a top. iso as well and $(A')^{-1} = (A^{-1})'$.

§5. Support and singular support of distributions

<u>Definition 5.1</u> $T \in \mathcal{D}'(\Omega)$, $\mathcal{U} \subset \Omega$ offen / open. Then

$$T \restriction \mathcal{U} := T \restriction \mathcal{D}(\mathcal{U}).$$

- In particular $T \restriction \mathcal{U} = 0 \iff T \restriction \mathcal{D}(\mathcal{U}) = 0$
$$\iff \forall \varphi \in \mathcal{D}(\mathcal{U}) : \langle T, \varphi \rangle = 0$$
- Similarly $T \restriction \mathcal{U}$ smooth $\iff T \restriction \mathcal{U} = T_\psi$ with $\psi \in C^\infty(\mathcal{U})$
$$\iff \exists \psi \in C^\infty(\mathcal{U})\ \forall \varphi \in \mathcal{D}(\mathcal{U})$$
$$\langle T, \varphi \rangle = \int_{\mathcal{U}} \psi \cdot \varphi$$

Such ψ is uniquely determined.

Definition 5.2 Let $T \in \mathcal{D}'(\Omega)$. Then

a) $\underline{\operatorname{supp} T} := \Omega \setminus \overbrace{\Big[\bigcup_{\substack{\Omega' \subset \Omega \text{ open} \\ \text{s.t. } T \restriction \Omega' = 0}} \Omega' \Big]}^{(\operatorname{supp} T)^c}$

i.e $x_0 \notin \operatorname{supp} T \Leftrightarrow \exists\, \mathcal{U}_{x_0} \subset \Omega$ open nbd : $\forall \varphi \in C_0^\infty(\mathcal{U})$: $T(\varphi) = 0$.

b) $\underline{\text{sing-supp}\, T} := \Omega \setminus \overbrace{\Big[\bigcup_{\substack{\Omega' \subset \Omega \text{ open} \\ \text{s.t. } T \restriction \Omega' \text{ smooth}}} \Omega' \Big]}^{(\operatorname{singsupp} T)^c}$

ie $x_0 \notin \operatorname{singsupp} T \Leftrightarrow \exists\, \mathcal{U}_{x_0} \subset \Omega$ open nbd, $\psi \in C^\infty(\mathcal{U})$ $\forall \varphi \in C_0^\infty(\mathcal{U})$:

$$T(\varphi) = \int_{\mathcal{U}} \psi \varphi .$$

Lemma 5.3 (Ü) Let $T \in \mathcal{D}'(\Omega)$, $\mathcal{U} = \bigcup_{i \in I} \mathcal{U}_i \subset \Omega$ open

(a) $\forall_i\ T \restriction \mathcal{U}_i = 0 \Rightarrow T \restriction \mathcal{U} = 0$

(b) $\forall_i\ T \restriction \mathcal{U}_i$ smooth $\Rightarrow T \restriction \mathcal{U}$ smooth.

Consequently:

> FACT: $\operatorname{singsupp} T \subseteq \operatorname{supp} T$ closed in Ω and
> - $T \restriction \Omega \setminus \operatorname{supp} T = 0$
> - $T \restriction \Omega \setminus \operatorname{singsupp} T$ smooth.

Examples 5.4
1) $\operatorname{supp} \delta_a = \operatorname{singsupp} \delta_a = \{a\}$
2) $f(x) = \begin{cases} \ln x, & x > 0 \\ 0, & x \leq 0 \end{cases}$ then

$\operatorname{supp} f = [0, \infty)$, $\operatorname{singsupp} f = \{0\}$.

Theorem 5.5 $\mathcal{E}'(\Omega) = \{ T \in \mathcal{D}'(\Omega) \mid \text{supp}\, T \text{ compact} \}$

Hence $\mathcal{E}'(\Omega)$ are indeed "distributions with compact support".

Proof: "$\supseteq$" We need to show the following points:

- T defines a linear mapping $\mathcal{E}(\Omega) = C^\infty(\Omega) \to \mathbb{C}$
- $T: \mathcal{E}(\Omega) \to \mathbb{C}$ is continuous ~~in the weak top.~~

Let $\psi \in C_0^\infty(\Omega)$ a cutoff-function with $\psi|_{\text{supp}\, T} \equiv 1$.
For each $\varphi \in \mathcal{E}(\Omega)$

$$\varphi = \psi\varphi + (1-\psi)\varphi$$

with $\text{supp}\,(1-\psi)\varphi \subset (\text{supp}\, T)^c$
and hence $T((1-\psi)\varphi) = 0$

Thus we may define $T: \mathcal{E}(\Omega) \to \mathbb{C},\ \varphi \mapsto T(\underbrace{\psi\varphi}_{\in \mathcal{D}(\Omega)})$
For continuity let $\varphi_n \xrightarrow{n\to\infty} \varphi$ in $\mathcal{E}(\Omega)$.

$$\Rightarrow \psi\varphi_n \xrightarrow{n\to\infty} \psi\varphi \text{ in } \mathcal{D}(\Omega)$$

$$\Rightarrow T(\psi\varphi_n) \to T(\psi\varphi), \text{ since } T \in \mathcal{D}'(\Omega)$$

ie $T' = T(\psi\,\cdot) : \mathcal{E}(\Omega) \to \mathbb{C}$ is cts ~~in the weak topology of $\mathcal{E}(\Omega)$~~. $\boxtimes$

"$\subseteq$". We know $\text{supp}\, T \subseteq \Omega$ closed $(T \in \mathcal{E}'(\Omega))$. Need to show

- $\text{supp}\, T$ is bounded
- $\text{supp}\, T$ is closed in $\mathring{\Omega}$.

Assume $\text{supp}\, T$ is unbded. (possible only if Ω is unbded)
Then there exist $(U_j)_{j\in\mathbb{N}}$ open nbds $U_j = B_\varepsilon(x_j)$, $\varepsilon > 0$ fixed with $(x_j) \subset \text{supp}\, T$ and distance between x_i, x_j always $> 2\varepsilon$. Choose for any $j \in \mathbb{N}$ some $\varphi_j \in C_0^\infty(U_j)$ with

$$T\varphi_j = 1.$$

Define $\varphi_N(x) := \sum_{j=1}^{N} \varphi_j(x)$, $(\varphi_N)_N \subset C^\infty(\Omega)$
and $\varphi_N \longrightarrow \varphi = \sum_{j=1}^{\infty} \varphi_j \in C^\infty(\Omega)$ (note that since
supports of φ_j are disjoint, this sum trivially converges pointwise)
converges in $\mathcal{E}(\Omega)$-topology, since each $K \subset \Omega$ cpt contains
only finitely many (x_j). Since $T \in \mathcal{E}'(\Omega)$;

$$T(\varphi_N) \xrightarrow{N \to \infty} T(\varphi), \text{ but } T(\varphi) = \sum T(\varphi_j) = \infty \quad \lightning$$

$\Rightarrow$ supp T is bded.

Assume there exists $x \in \partial\Omega$ st. $\forall \varepsilon > 0 : B_\varepsilon(x) \cap \operatorname{supp} T \neq \emptyset$.
Then there exists $(\mathcal{U}_j)_{j \in \mathbb{N}}$ with $\mathcal{U}_j = B_{\varepsilon_j}(x_j)$ where

$$(x_j) \subset \operatorname{supp} T, \; x_j \longrightarrow x, \; \varepsilon_j \longrightarrow 0.$$
$$\text{and } \mathcal{U}_i \cap \mathcal{U}_j = \emptyset \text{ for } i \neq j.$$

Choose φ_j as before. Same argument $\lightning$ $\blacksquare$

Sometimes following characterization is useful

<u>Proposition 5.6</u> (without proof)

Let $T \in \mathcal{D}'(\Omega)$, such that $\forall \varphi \in \mathcal{D}(\Omega)$ with supp $\varphi \subset K$ cpt (for any fixed cpt K)

$$|T(\varphi)| \leq C \cdot \|\varphi\|_{C^k(K)}$$

and supp $T = \{x_0\} \in \Omega$. Then for any $\varphi \in C_0^\infty(\Omega)$

$$T(\varphi) = \sum_{|\alpha| < k} a_\alpha \cdot (\partial_x^\alpha \varphi)(x_0)$$