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The Randomized Kaczmarz Method with Mismatched Adjoint

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Abstract This paper investigates the randomized version of the Kaczmarz method to solve linear systems in the case where the adjoint of the system matrix is not exact—a situation we refer to as "mismatched adjoint". We show that the method may still converge both in the over- and underdetermined consistent case under appropriate conditions, and we calculate the expected asymptotic rate of linear convergence. Moreover, we analyze the inconsistent case and obtain results for the method with mismatched adjoint as for the standard method. Finally, we derive a method to compute optimized probabilities for the choice of the rows and illustrate our findings with numerical examples.

Keywords randomized algorithms · Kaczmarz method · linear convergence

Mathematics Subject Classification (2000) 65F10 · 68W20 · 15A24

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1 Introduction

In this paper we consider the solution of linear systems

$$Ax = b \tag{1.1}$$

with row-action methods, i.e methods that only use single rows of the system in each step. This is beneficial, for example, in situations where the full system is too large to store or keep in memory. Probably the first method of this type is the Kaczmarz method where each step consists of a projection onto a hyperplane given by the solution space of a single row. If a^T is a row vector of the system and the corresponding entry on the right hand side is (with slight abuse of notation) b, then the orthogonal projection of a given vector x onto the solution space of $\langle a\,,x\rangle = b$ is

$$x - \frac{\langle a\,,x\rangle - b}{\|a\|^2}\cdot a.$$

Thus, one updates the current vector x in the direction of a which is the corresponding column of A^T. A question that has been motivated by the use of the Kaczmarz method in tomographic reconstruction (where it is known under the name *algebraic reconstruction technique* (ART), [4], see also [6]) is: *Will the method still converge, if we do not use A^T as the adjoint but a different matrix V^T ?* In tomographic reconstruction, the linear operator A models the "forward projection" operation, which maps an object's density to a set of measured line integrals. The adjoint map A^T, however, also has a physical interpretation: This map is called "backprojection" and, roughly speaking, "distributes the values along lines through the measurement volume". Since both A and A^T have their own physical significance, their corresponding maps are often implemented by different means. For example, [2] proposes and discusses several methods for the implementation of the backprojection method and shows that special methods compare favorably with respect to reconstruction quality. In [12], the authors discuss the use of mismatched projection pairs, for the purposes of improved computational efficiency when using the Landweber algorithm for reconstruction. Hence, one does not always use the actual adjoint, but a different map, and we refer to this situation as using a "mismatched adjoint".

The goal of this paper is to analyze the convergence behavior of the randomized Kaczmarz method with mismatched adjoint.

Notation. For a vector $x \in \mathbb{R}^n$ we denote by $\mathrm{Diag}(x)$ the $n \times n$ matrix with the entries of x on the diagonal and, similar to the MATLAB operator, for an $n \times n$ matrix A, $\mathrm{Diag}(A)$ denotes the n vector of diagonal entries of A. For two $A, B \in \mathbb{R}^{n\times n}$ we use the inner product $\langle A\,,B\rangle = \mathrm{trace}(A^T B)$ (which induces the Frobenius norm) and we see that the map $\mathrm{Diag} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ fulfills

$$\langle \mathrm{Diag}(x)\,,A\rangle = \mathrm{trace}(\mathrm{Diag}(x)A) = \sum_{i=1}^{n} x_i a_{ii} = \langle x\,,\mathrm{Diag}(A)\rangle,$$

i.e. the linear map Diag is "self adjoint" in a slightly weird way.

For a real and symmetric matrix M we denote by $\lambda_{\min}(M)$ the smallest eigenvalue of M and for general (non-symmetric) real square matrices M we denote by $\rho(M)$ its *spectral radius*, i.e. the largest absolute value of its eigenvalues.

2 The overdetermined consistent case

The Kaczmarz method is known to converge for any consistent linear system, but the speed of convergence is hard to quantify since it depends on the ordering of the rows. This is notably different for the *randomized Kaczmarz method* as shown in [10]: If the rows are chosen independently at random the method converges linearly. To fix notation, let $A = (a_i^T)_{i=1,\dots,m} \in \mathbb{R}^{m\times n}$ with $m \geq n$ and vectors $a_i \in \mathbb{R}^n$ and $V = (v_i^T)_{i=1,\dots,m}$, with vectors $v_i \in \mathbb{R}^n$. Moreover let $p_i > 0$, $i \in \{1,\dots,m\}$ denote a probability distribution on the set of indices of the rows, i.e., p_i is the probability to choose the i-th row for the next step. We discuss several choices of these probabilities later in Remark 2.4 and methods to determine optimal probabilities (in a certain sense) in Section 5.

Algorithm 1 Randomized Kaczmarz with Mismatched Adjoint

Input: starting point $x_0 \in \mathbb{R}^n$, probabilities $p_i > 0$, $i \in \{1,\dots,m\}$, matrices A and V

1: initialize $k = 0$
2: **repeat**
3: choose an index $i_k = i \in \{1,\dots,m\}$ at random with probability p_i
4: update $x_{k+1} = x_k - \frac{\langle a_{i_k}, x_k\rangle - b_{i_k}}{\langle a_{i_k}, v_{i_k}\rangle} \cdot v_{i_k}$
5: increment $k = k+1$
6: **until** a stopping criterion is satisfied

The algorithm we consider in this work is the randomized Kaczmarz method with mismatched adjoint, abbreviated RKMA, and is given in Algorithm 1. The difference to the standard randomized Kaczmarz method is that the usual projection step $x_{k+1} = x_k - \frac{\langle a_{i_k}, x_k\rangle - b_{i_k}}{\|a_{i_k}\|^2} \cdot a_{i_k}$ is replaced by $x_{k+1} = x_k - \frac{\langle a_{i_k}, x_k\rangle - b_{i_k}}{\langle a_{i_k}, v_{i_k}\rangle} \cdot v_{i_k}$. This results in $\langle x_{k+1}, a_{i_k}\rangle = b_{i_k}$, i.e., the next iterate x_{k+1} is on the hyperplane defined by the i_k-th equation of the system, but since v_{i_k} is not orthogonal to this hyperplane, this is an *oblique* projection, instead of an orthogonal projection as it would be in the original Kaczmarz method (see Figure 2.1).

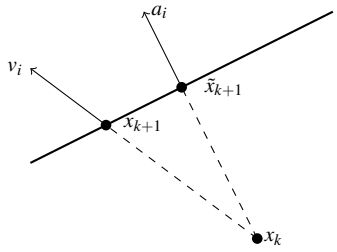


Fig. 2.1 Oblique projection x_{k+1} of x_k onto the hyperplane $\{x \mid \langle a_i, x\rangle = b_i\}$. The orthogonal projection is $\tilde{x}_{k+1}$.

First we state a result on the expected outcome of one step of RKMA. A similar result has been observed earlier in the case with no mismatch, see e.g. [10], [8,

Lemma 2.2]) or [13, Lemma 3.6]). In the following, we generally assume that the rows of A and V fulfill $\langle a_i\,,v_i\rangle \neq 0$ and, without loss of generality, that $\langle a_i\,,v_i\rangle > 0$.[1]

Lemma 2.1 *Let $\hat{x}$ fulfill $A\hat{x} = b$, x be arbitrary and $x^+ = x - \frac{\langle a_i\,,x\rangle - b_i}{\langle a_i\,,v_i\rangle}\cdot v_i$ be the oblique projection onto the hyperplane $\{x \mid \langle a_i\,,x\rangle = b_i\}$. Further we let $p_i > 0$, $i = 1,\dots,m$, be probabilities and denote $D := \operatorname{Diag}\left(\frac{p_i}{\langle a_i\,,v_i\rangle}\right)$ and $S := \operatorname{Diag}\left(\frac{\|v_i\|^2}{\langle a_i\,,v_i\rangle}\right)$. If i is randomly chosen with probability p_i (i.e., x^+ is a random variable) then it holds that*

$$\mathbb{E}(x^+ - \hat{x}) = (I - V^T DA)(x - \hat{x}) \tag{2.1}$$

and if

$$\lambda := \lambda_{\min}\left(V^T DA + A^T DV - A^T SDA\right) > 0 \tag{2.2}$$

is fulfilled, it holds that

$$\mathbb{E}(\|x^+ - \hat{x}\|^2) \leq (1-\lambda)\cdot\|x - \hat{x}\|^2$$

(where both expectations are with respect to the probabilities p_i).

Proof Since $b_i = \langle a_i\,,\hat{x}\rangle$, the expectation $\mathbb{E}(x^+ - \hat{x})$ is

$$\begin{aligned}\mathbb{E}(x^+ - \hat{x}) &= \sum_{i=1}^m p_i \cdot \left(x - \frac{\langle a_i\,,x\rangle - b_i}{\langle a_i\,,v_i\rangle}\cdot v_i\right) - \hat{x}\\ &= x - \sum_{i=1}^m p_i \cdot \frac{\langle a_i\,,x-\hat{x}\rangle}{\langle a_i\,,v_i\rangle}\cdot v_i - \hat{x}\\ &= x - \hat{x} - \sum_{i=1}^m \tfrac{p_i}{\langle a_i\,,v_i\rangle}\cdot v_i a_i^T (x-\hat{x}),\end{aligned}$$

where we deliberately wrote the outer product $v_i a_i^T$ to make it apparent that (2.1) follows from here. To calculate the expectation of the squared norm we calculate

$$\begin{aligned}\|x^+ - \hat{x}\|^2 =& \|x-\hat{x}\|^2 - 2\cdot\frac{\langle a_i\,,x-\hat{x}\rangle\cdot\langle v_i\,,x-\hat{x}\rangle}{\langle a_i\,,v_i\rangle}\\ &+ \frac{\left(\langle a_i\,,x-\hat{x}\rangle\right)^2}{\left(\langle a_i\,,v_i\rangle\right)^2}\cdot\|v_i\|^2.\end{aligned} \tag{2.3}$$

Taking the expectation gives

$$\begin{aligned}\mathbb{E}(\|x^+ - \hat{x}\|^2) =& \|x-\hat{x}\|^2\\ &- \sum_{i=1}^m p_i\cdot 2\cdot\frac{\langle a_i\,,x-\hat{x}\rangle\cdot\langle v_i\,,x-\hat{x}\rangle}{\langle a_i\,,v_i\rangle}\\ &+ \sum_{i=1}^m p_i\cdot\frac{\left(\langle a_i\,,x-\hat{x}\rangle\right)^2}{\left(\langle a_i\,,v_i\rangle\right)^2}\cdot\|v_i\|^2.\end{aligned}$$

[1] If $\langle a_i\,,v_i\rangle = 0$, Algorithm 1 is not defined and in the case of $\langle a_i\,,v_i\rangle < 0$, the probabilities p_i below in Rmeark 2.4 would not be non-negative. However, in the case $\langle a_i\,,v_i\rangle < 0$ we could switch the sign of the v_is, and the expressions in (2.1) and (2.2) would not change).

By the definition of D and S the right hand side can be written as

$$\begin{aligned} &\|x-\hat{x}\|^2 - \langle x-\hat{x}, (2V^TDA - A^TSDA)(x-\hat{x})\rangle \\ &= \|x-\hat{x}\|^2 - \langle x-\hat{x}, (2V^T - A^TS)DA(x-\hat{x})\rangle \end{aligned} \tag{2.4}$$

and hence, we aim to bound $\langle x-\hat{x}, (2V^T - A^TS)DA(x-\hat{x})\rangle$ from below. More precisely, we want

$$\langle x-\hat{x}, (2V^T - A^TS)DA(x-\hat{x})\rangle \geq \lambda \cdot \|x-\hat{x}\|^2$$

and this is the case if and only if

$$\langle x-\hat{x}, ((2V^T - A^TS)DA - \lambda I)(x-\hat{x})\rangle \geq 0.$$

Since we have $2\langle z, V^TDAz\rangle = \langle z, (V^TDA + A^TDV)z\rangle$ for all z, this is equivalent to

$$\langle x-\hat{x}, (V^TDA + A^TDV - A^TSDA - \lambda I)(x-\hat{x})\rangle \geq 0$$

and this is ensured if

$$\lambda_{\min}(V^TDA + A^TDV - A^TSDA) \geq \lambda.$$

Hence, if (2.2) is fulfilled, we obtain the estimate

$$\mathbb{E}(\|x_{k+1}-\hat{x}\|^2) \leq (1-\lambda)\cdot\|x-\hat{x}\|^2.$$

□

Equation (2.1) shows that $\|\mathbb{E}(x^+-\hat{x})\|^2 \leq \|I-V^TDA\|^2\|x-\hat{x}\|^2$. Recall that $\rho(M) \leq \|M\|$ for asymmetric matrices M, and note that the above inequality is not true, if we replace the norm by the spectral radius. Due to Jensen's inequality we generally have $\|\mathbb{E}(x^+-\hat{x})\|^2 \leq \mathbb{E}(\|x^+-\hat{x}\|^2)$ and Lemma 2.1 provides different estimates for both quantities.

Iterating the previous lemma, we obtain the convergence result:

Theorem 2.2 *Assume that the assumptions of Lemma 2.1 are fulfilled and denote by x_k the iterates of Algorithm 1.*

If $\rho(I-V^TDA) < 1$ then x_k converges in expectation to $\hat{x}$,

$$\mathbb{E}(x_k-\hat{x}) \to 0 \quad \textit{for} \quad k\to\infty,$$

moreover, it holds that

$$\|\mathbb{E}(x_k-\hat{x})\| \leq \|I-V^TDA\|^k\|x_0-\hat{x}\|.$$

If condition (2.2) *is fulfilled then it holds that*

$$\mathbb{E}\left[\|x_{k+1}-\hat{x}\|^2\right] \leq (1-\lambda)\cdot\mathbb{E}\left[\|x_k-\hat{x}\|^2\right].$$

Proof The first claim follows from Lemma 2.1 and the well known fact that $(I - V^T DA)^k \to 0$ if the spectral radius of $I - V^T DA$ is smaller than one (see, e.g., [3, Theorem 11.2.1]). The second claim is also immediate from the previous lemma.

Finally, we get for expectation with respect to i_k (conditional on $i_0, \dots, i_{k-1}$)

$$\mathbb{E}\left[\|x_{k+1} - \hat{x}\|^2 \,\middle|\, i_0, \dots, i_{k-1}\right] \leq (1-\lambda)\|x_k - \hat{x}\|^2$$

Now we consider all indices $i_0, \dots, i_k$ as random variables with values in $\{1, \dots, m\}$, and take the full expectation on both sides to get the assertion. □

Here are some remarks on the result:

Remark 2.3 *Since eigenvalues depend continuously on perturbations, both condition* (2.2) *and* $\rho(I - V^T DA) < 1$ *are fulfilled for* $V \approx A$. *Note that* $\|I - V^T DA\| = \rho(I - V^T DA)$ *does hold for* $V = A$ *and is generally not true otherwise. It may even be the case that* $\|I - V^T DA\| > 1$ *while* $\rho(I - V^T DA) < 1$.

Remark 2.4 (Relation to the result of Strohmer and Vershynin) *Note that Theorem 2.2 contains the result of Strohmer and Vershynin [10] as a special case: Take* $V = A$ *and the probabilities* p_i *proportional to the squared row-norms, i.e.,* $p_i = \frac{\|a_i\|^2}{\|A\|_F^2}$. *Then we have*

$$D = \operatorname{Diag}\left(\tfrac{p_i}{\langle a_i, v_i\rangle}\right) = \tfrac{1}{\|A\|_F^2} \cdot I \quad \textit{and} \quad S = \operatorname{Diag}\left(\tfrac{\|v_i\|^2}{\langle a_i, v_i\rangle}\right) = I$$

and hence we get

$$\lambda = \frac{\lambda_{\min}(A^T A)}{\|A\|_F^2} = \frac{\sigma_{\min}(A)}{\|A\|_F^2}$$

(where $\sigma_{\min}(A)$ *denotes the smallest singular value of* A*) as in [10].*

To get a similarly simple expression for the convergence rate of the method with mismatch we set

$$p_i = \frac{\langle a_i, v_i\rangle}{\|A\|_V^2}, \quad \textit{with} \quad \|A\|_V^2 = \sum_i \langle a_i, v_i\rangle.$$

This leads to

$$D = \tfrac{1}{\|A\|_V^2} \cdot I$$

and thus, from (2.1),

$$\|\mathbb{E}(x_{k+1} - \hat{x})\| \leq \|I - \tfrac{V^T A}{\|A\|_V^2}\| \, \|x_k - \hat{x}\| = \sigma_{\max}(I - \tfrac{V^T A}{\|A\|_V^2}) \cdot \|x_k - \hat{x}\|.$$

However, in general the contraction factor does not simplify to $1 - \frac{\sigma_{\min}(V^T A)}{\|A\|_V^2}$ *as it would in the case with no mismatch.*

We also get

$$\mathbb{E}(\|x_{k+1} - \hat{x}\|) \leq \left(1 - \tfrac{\lambda_{\min}(V^T A + A^T V - A^T S A)}{\|A\|_V^2}\right)^{1/2} \|x_k - \hat{x}\|$$

for the expectation of the error.

Remark 2.5 (Asymptotic convergence rate and expected improvement in norm) *The above theorem states that the RKMA method has the asymptotic convergence rate of*

$$\rho(I - V^T DA) \tag{2.5}$$

(in expectation), however, the expected improvement of the squared error, i.e. $\mathbb{E}(\|x_k - \hat{x}\|^2)$ *in every iteration is*

$$\begin{aligned}(1 - \lambda_{\min}(V^T DA + A^T DV - A^T SDA)) &= \rho(I - V^T DA - A^T DV + A^T SDA) \\ &= \|I - V^T DA - A^T DV + A^T SDA\|.\end{aligned} \tag{2.6}$$

Using the spectral norm we can also estimate

$$\|\mathbb{E}(x_{k+1} - \hat{x})\| = \|(I - V^T DA)(x_k - \hat{x})\| \leq \|I - V^T DA\| \cdot \|x_k - \hat{x}\|.$$

We can express this norm by the spectral radius as

$$\|I - V^T DA\| = \rho(I - V^T DA - A^T DV + A^T DVV^T DA). \tag{2.7}$$

Note that all three expressions in (2.5), (2.6) (2.7) *are equal in the case of* $V = A$*, but for the mismatched case, they are in general different. Numerically it seems like* (2.5) $\leq$ (2.7) $\leq$ (2.6)*, but we do not have a proof for this.*

Remark 2.6 (Different possibilities for stepsizes) *We could consider the slightly more general iteration*

$$x_{k+1} = x_k - \omega_{i_k} \cdot (\langle a_{i_k}, x_k \rangle - b_{i_k}) \cdot v_{i_k}$$

with a steplength ω_{i_k}*. The iteration in Algorithm 1 uses* $\omega_i = \langle a_i, v_i \rangle^{-1}$*, but there are other meaningful choices:*

- *As for the case with no mismatch, one could take* $\omega_{i_k} = \|a_{i_k}\|^{-2}$*, but this would not imply* $\langle x_{k+1}, a_{i_k} \rangle = b_{i_k}$*. Similarly,* $\omega_i = \|v_{i_k}\|^{-2}$ *does not imply* $\langle x_{k+1}, v_{i_k} \rangle = b_{i_k}$*.*
- *The choice* $\omega_{i_k} = \frac{\langle x_k, v_{i_k} \rangle - b_{i_k}}{(\langle x_k, a_{i_k} \rangle - b_{i_k})\|v_i\|^2}$ *implies that* $\langle x_{k+1}, v_{i_k} \rangle = b_{i_k}$*.*

Although none of these cases guarantees that the iterates solve one of the equations of the linear system $Ax = b$*, one can still deduce that iterates converge to the solution of this system of equalities. The result of Theorem 2.2 can also be derived for this slightly more general iteration and the respective condition for linear convergence with contraction factor* $(1 - \lambda)$ *is that*

$$\lambda := \lambda_{\min}\left(V^T DA + A^T DV - A^T SDA\right) > 0$$

with

$$D = \mathrm{Diag}(p_i \omega_i), \qquad S = \mathrm{Diag}(\omega_i \|v_i\|^2).$$

Experiments show that other probabilities than $p_i = \|a_i\|^2 / \|A\|_F^2$ in the case $V = A$ or $p_i = \langle a_i, v_i \rangle / \|A\|_V^2$ in the mismatched case frequently lead to faster convergence. This should not be surprising as one could scale the rows of system $Ax = b$ arbitrarily by multiplying with a diagonal matrix which leaves the solution unchanged, but leads to arbitrary row-norms of the scaled system. In this sense, the row-norms do not reflect the geometry of the arrangements of hyperplanes. We will come back to the problem of selecting probabilities in Section 5.

3 Inconsistent overdetermined systems

Now we consider the inconsistent case, i.e., we do not assume that the overdetermined system has a solution. This case has been treated in [8] for the case $V = A$. We model an additive error and assume that the right hand side is $b + r$ with $b \in \mathrm{rg} A$.

Theorem 3.1 *Denote by $\hat{x}$ the unique solution of $Ax = b$ and let x_k denote the iterates of Algorithm 1 where the right hand side is $b + r$. With $M = (I - V^T DA)$ it holds that*

$$\mathbb{E}(x_k - \hat{x}) = M^k (x_0 - \hat{x}) + \sum_{l=0}^{k-1} M^l V^T D r. \tag{3.1}$$

Moreover, with λ defined in (2.2)*, we have*

$$\mathbb{E}(\|x_k - \hat{x}\|^2) \leq (1 - \tfrac{\lambda}{2})^k \cdot \|x_0 - \hat{x}\|^2 + \tfrac{2(2-\lambda)}{\lambda^2} \cdot \gamma^2 \tag{3.2}$$

with $\gamma := \max_i \frac{|r_i| \cdot \|v_i\|}{|\langle a_i, v_i \rangle|}$.

Proof For the iterate x_k we denote by $\tilde{x}_{k+1}$ the oblique projection onto the "true hyperplane" $H = \{x \mid \langle a_i, x \rangle = b_i\}$, i.e., $\tilde{x}_{k+1} = x_k - \frac{\langle a_i, x_k \rangle - b_i}{\langle v_i, a_i \rangle} \cdot v_i$. Then it holds that

$$x_{k+1} - \hat{x} = \tilde{x}_{k+1} - \hat{x} + \tfrac{r_i}{\langle a_i, v_i \rangle} \cdot v_i.$$

For one step of the method we get (taking the expectation with respect to the random variable i_{k+1})

$$\mathbb{E}(x_{k+1} - \hat{x}) = \mathbb{E}(\tilde{x}_{k+1} - \hat{x}) + \mathbb{E}(\tfrac{r_{i_k}}{\langle a_{i_k}, v_{i_k} \rangle} v_{i_k}) = (I - V^T DA)(x_k - \hat{x}) + V^T D r.$$

The formula for $\mathbb{E}(x_k - \hat{x})$ (with the expectation with respect to all indices $i_0, \ldots, i_k$) follows by induction.

Moreover, we get

$$\begin{aligned} \|x_{k+1} - \hat{x}\|^2 &= \|\tilde{x}_{k+1} - \hat{x}\|^2 + 2 \tfrac{r_i}{\langle a_i, v_i \rangle} \cdot \langle \tilde{x}_{k+1} - \hat{x}, v_i \rangle + \tfrac{r_i^2}{\langle a_i, v_i \rangle^2} \cdot \|v_i\|^2 \\ &\leq \|\tilde{x}_{k+1} - \hat{x}\|^2 + 2 \tfrac{r_i}{\langle a_i, v_i \rangle} \cdot \langle \tilde{x}_{k+1} - \hat{x}, v_i \rangle + \gamma^2. \end{aligned} \tag{3.3}$$

Now we use Cauchy-Schwarz and Young with $\varepsilon > 0$ (i.e. $2ab \leq \varepsilon a^2 + b^2/\varepsilon$) to get

$$\begin{aligned} \|x_{k+1} - \hat{x}\|^2 &\leq \|\tilde{x}_{k+1} - \hat{x}\|^2 + 2 \|\tilde{x}_{k+1} - \hat{x}\| \cdot \tfrac{r_i}{\langle a_i, v_i \rangle} \cdot \|v_i\| + \gamma^2 \\ &\leq (1 + \varepsilon) \cdot \|\tilde{x}_{k+1} - \hat{x}\|^2 + (1 + \tfrac{1}{\varepsilon}) \cdot \gamma^2. \end{aligned}$$

Applying Lemma 2.1 we get

$$\mathbb{E}(\|x_{k+1} - \hat{x}\|^2) \leq (1 + \varepsilon) \cdot (1 - \lambda) \cdot \|x_k - \hat{x}\|^2 + (1 + \tfrac{1}{\varepsilon}) \cdot \gamma^2.$$

Recursively we obtain

$$\mathbb{E}(\|x_k - \hat{x}\|^2) \leq \Big((1 + \varepsilon) \cdot (1 - \lambda)\Big)^k \cdot \|x_0 - \hat{x}\|^2 + \sum_{j=0}^{k-1} \Big((1 + \varepsilon) \cdot (1 - \lambda)\Big)^j \cdot (1 + \tfrac{1}{\varepsilon}) \cdot \gamma^2.$$

Now we choose $\varepsilon = \frac{\lambda}{2(1-\lambda)}$, observe that

$$(1-\lambda)\cdot(1+\varepsilon) = 1-\tfrac{\lambda}{2} \quad \text{and} \quad (1+\tfrac{1}{\varepsilon}) = \tfrac{2}{\lambda}-1$$

and get

$$\begin{aligned}\mathbb{E}(\|x_k-\hat{x}\|^2) &\leq (1-\tfrac{\lambda}{2})^k\cdot\|x_0-\hat{x}\|^2 + \sum_{j=0}^{k-1}(1-\tfrac{\lambda}{2})^{j+1}(\tfrac{2}{\lambda}-1)\cdot\gamma^2 \\ &\leq (1-\tfrac{\lambda}{2})^k\cdot\|x_0-\hat{x}\|^2 + \tfrac{2(2-\lambda)}{\lambda^2}\cdot\gamma^2\end{aligned}$$

which proves the claim. □

The first equation in Theorem 3.1 shows that the iteration of RKMA will reach a final error of the order of $\|\sum_{l=0}^{\infty} M^l V^T Dr\| = \|(I-M)^{-1}V^T Dr\| = \|(V^T DA)^{-1}V^T Dr\|$ if $\rho(M) < 1$.

The final estimate in Theorem 3.1 is a little worse than the respective result in [8, Theorem 2.1]. For one, the factor in front of $\|x_0-\hat{x}\|^2$ is only $(1-\lambda/2)$ instead of $(1-\lambda)$ (since λ corresponds to $1/R$ in [8] in the case of no mismatch and "standard" probabilities $p_i = \|a_i\|_2^2/\|A\|_F^2$) and also, the constant in front of γ is larger. This estimate can be improved a little bit by noting that $\langle \tilde{x}_{k+1}-\hat{x}, a_i\rangle = 0$ and thus, we can estimate the right hand side in (3.3) by

$$(1+\varepsilon)\|\tilde{x}_{k+1}-\hat{x}\|^2 + (1+\tfrac{\delta^2}{\varepsilon})\|\gamma\|^2$$

with $\delta = \max_i \frac{\|v_i-a_i\|}{\|v_i\|}$. In case $\delta = 0$ (i.e., no mismatch) we can even take $\varepsilon = 0$ and recover the result from [8]. For $\delta > 0$ we get the final estimate

$$\mathbb{E}(\|x_k-\hat{x}\|^2) \leq (1-\tfrac{\lambda}{2})^k\cdot\|x_0-\hat{x}\|^2 + \tfrac{2\lambda(1-2\delta^2)+4\delta^2}{\lambda^2}\cdot\gamma^2$$

which gives the result from Theorem 3.1 for $\delta = 1$ but is better for $\delta < 1$.

4 Underdetermined systems

Now we consider the underdetermined case, i.e., the case where $m < n$, but we will still assume full row rank of A and V. In the case of no mismatch, linear convergence has been proven for the probabilities $p_i = \|a_i\|^2/\|A\|_F^2$ in [7]. In this case, convergence does not follow from Theorem 2.2: On the one hand, the matrix $V^T DA + A^T DV$ is never positive definite, so λ from (2.2) is always zero. On the other hand, $V^T DA$ always has a non-trivial kernel, and thus, $I - V^T DA$ always has spectral radius equal to one. However, the iteration often converges in practice and this is due to the following simple observation: All the iterates x_k of Algorithm 1 are in $\operatorname{rg} V^T$ if the starting point x_0 is there. So, if the equation $Ax = b$ has a solution $\hat{x}$ in $\operatorname{rg} V^T$, then all vectors $x_k - \hat{x}$ are also in the range.

Inspecting the proof of Lemma 2.1 we note that the constant λ that needs to be positive to guarantee improvement in each step, is in fact not the smallest eigenvalue of $V^T DA + V^T DA - A^T DSA$ but the smallest eigenvalue of this matrix when restricted

to the range of V^T. More explicitly, let $Z \in \mathbb{R}^{n\times m}$ be a matrix whose columns form on orthonormal basis of $\operatorname{rg} V^T$. So, the term in (2.4) can also be written as

$$\begin{aligned}&\|x_k - \hat{x}\|^2 - \langle ZZ^T(x_k - \hat{x}), (2V^TDA - A^TDSA)ZZ^T(x_k - \hat{x})\rangle \\ &= \|x_k - \hat{x}\|^2 - \langle Z^T(x_k - \hat{x}), Z^T(2V^TDA - A^TDSA)ZZ^T(x_k - \hat{x})\rangle.\end{aligned}$$

Consequently, we need an estimate of the form

$$\langle Z^T(x_k - \hat{x}), Z^T(2V^TDA - A^TDSA)ZZ^T(x_k - \hat{x})\rangle \geq \lambda \cdot \|x_k - \hat{x}\|^2$$

and, since $\|Z^T(x_k - \hat{x})\|^2 = \|x_k - \hat{x}\|^2$, this is fulfilled for

$$\lambda = \lambda_{\min}(Z^T(V^TDA + A^TDV - A^TDSA)Z).$$

Similarly, the convergence of $\mathbb{E}(x_k - \hat{x})$ is equivalent to the convergence of $\mathbb{E}(Z^T(x_k - \hat{x}))$, and it holds that

$$\begin{aligned}\mathbb{E}(Z^T(x_{k+1} - \hat{x})) &= Z^T(I - V^TDA)(x_k - \hat{x}) \\ &= Z^T(I - V^TDA)ZZ^T(x_k - \hat{x}) \\ &= (I - Z^TV^TDAZ)Z^T(x_k - \hat{x}).\end{aligned}$$

Finally, note that the system $Ax = b$ has only one solution that lies in $\operatorname{rg} V^T$ if AV^T is non-singular.

Thus, we have proved the following theorem:

Theorem 4.1 *Consider the consistent system* (1.1) *with $A, V \in \mathbb{R}^{m\times n}$ for $m \leq n$ both with full row rank such that AV^T is non-singular. Furthermore let the columns of Z be an orthonormal basis for $\operatorname{rg} V^T$ and let $p \in \mathbb{R}^m$ with $p_i \geq 0$ and $\sum_i p_i = 1$ and set $D := \operatorname{Diag}\left(\frac{p_i}{\langle a_i, v_i\rangle}\right)$ and $S := \operatorname{Diag}\left(\frac{\|v_i\|^2}{\langle a_i, v_i\rangle}\right)$. Then it holds:*

1. *The system $Ax = b$ has exactly one solution $\hat{x}$ that lies in $\operatorname{rg} V^T$.*
2. *If $x_0 \in \operatorname{rg} V^T$ and $\rho(I - Z^TV^TDAZ) < 1$, then the iterates of Algorithm 1 fulfill*

$$\mathbb{E}(x_k - \hat{x}) \to 0 \quad \textit{for} \quad k \to \infty.$$

3. *If $x_0 \in \operatorname{rg} V^T$ and*

$$\lambda := \lambda_{\min}\left(Z^T(V^TDA + A^TDV - A^TSDA)Z\right) > 0 \tag{4.1}$$

is fulfilled, then it holds that

$$\mathbb{E}\left[\|x_{k+1} - \hat{x}\|^2\right] \leq (1 - \lambda) \cdot \mathbb{E}\left[\|x_k - \hat{x}\|^2\right].$$

This result has the following practical implication: If one can measure the quantity x by linear measurements, encoded by the vectors a_i, but only has less measurements available than degrees of freedom in x, it is beneficial to use a mismatched adjoint V with rows v_i^T such that the v_i are close to the vectors a_i (such that the convergence condition is fulfilled), but which also ensure that x is in the range of the vectors v_i. Mismatched forward/back projection models in CT provide a useful example to illustrate this result. Forward projection in CT is often implemented using

a ray-tracing algorithm known as Siddon's method [9]. This algorithm models line integration and has the benefit of being computationally efficient and amenable to parallelization; however, it does not model the finite width of the detector bin. This can lead to Moire pattern artifacts when using a matched forward/back-projection pair if the image pixel size is smaller than the detector bin size [2]. Mismatched projector pairs — in which the backprojection operator models the finite detector bin width — are often used to avoid these artifacts. We illustrate how RKMA can be used in this manner in Section 6.

5 Optimizing the probabilities

In the case of exact adjoint, a common choice for the probabilities p_i is to use $p_i = \|a_i\|^2/\|A\|_F^2$ which leads to the simple expression $\lambda = \lambda_{\min}(A^TA)/\|A\|_F^2$. However, numerical experiments show that this vector p of probabilities does not lead to the best performance in practice. This is of no surprise: For any diagonal matrix $W = \mathrm{Diag}(w_i)$ one can consider the problem $WAx = Wb$ which has different row norms, while each Kaczmarz iteration stays exactly the same. This shows that the choice of probabilities based on the norms of the rows is in some sense arbitrary. In [1] the authors proposed a method to find the smallest contraction factor of the method by minimizing the largest eigenvalue of an auxiliary matrix of size $\mathbb{R}^{n^2\times n^2}$. Here we present a different method that also works for the case of mismatched adjoint.

Theorem 2.2 states that the asymptotic convergence rate is given by $\rho(I-V^TDA)$, while the expected improvement in each step is either expressed by $1-\lambda_{\min}(V^TDA + A^TDV - A^TSDA)$ or $\|I-V^TDA\|$ (recall that $D = \mathrm{Diag}(p_i/\langle a_i, v_i\rangle)$ and $S = \mathrm{Diag}(s_i)$ with $s_i = \|v_i\|^2/\langle a_i, v_i\rangle$). One would like to choose p (i.e. D) in such a way that these quantities are as small as possible. Numerically, we observe that the asymptotic rate is indeed quite tight, while the expected improvement is only a loose estimate in the case of mismatched adjoint. However, the numerical radius of a non-symmetric matrix is not easily characterized and is neither a convex, nor concave function of the entries of the matrix. The minimal eigenvalue of a symmetric matrix, on the other hand, is characterized by a minimization problem and it will turn out, that $\lambda_{\min}$ is indeed a concave function in p. Also, the spectral norm is convex and thus, the function $\|I-V^TDA\|$ is also convex in p. We therefore aim to choose p such that $\lambda_{\min}$ is maximized or $\|I-V^TDA\|$ is minimized, i.e. we aim to solve

$$\max_p \ \lambda_{\min}(V^TDA + A^TDV - A^TSDA), \quad \text{s.t.} \quad \sum_{i=1}^m p_i = 1, \quad p \geq 0. \tag{5.1}$$

or

$$\min_p \ \|I-V^TDA\|, \quad \text{s.t.} \quad \sum_{i=1}^m p_i = 1, \quad p \geq 0. \tag{5.2}$$

5.1 Maximizing $\lambda_{\min}$

The super-gradient of the objective functional in (5.1) is given by the next lemma:

Lemma 5.1 *The function $f(p) = \lambda_{\min}(V^T DA + A^T DV - A^T SDA)$ is concave. A super-gradient at p is given by*

$$\frac{\partial \lambda_{\min}}{\partial p} = \left(\frac{\langle 2v_i - s_i a_i \, , x \rangle \langle a_i \, , x \rangle}{\langle a_i \, , v_i \rangle} \right)_{i=1,\dots,m}$$

where x is an eigenvector of $V^T DA + A^T DV - A^T SDA$ corresponding to the smallest eigenvalue.

Proof By the min-max principle for eigenvalues of symmetric matrices, we have

$$\begin{aligned} \lambda_{\min}(V^T DA + A^T DV - A^T SDA) &= \min_{\|x\|=1} \langle (V^T DA + A^T DV - A^T SDA)x \, , x \rangle \\ &= \min_{\|x\|=1} \langle DAx \, , (2V - SA)x \rangle \\ &= \min_{\|x\|=1} \sum_{i=1}^{m} p_i \frac{\langle 2v_i - s_i a_i \, , x \rangle \langle a_i \, , x \rangle}{\langle a_i \, , v_i \rangle}. \end{aligned}$$

This shows that f is a minimum over linear functions in p, and hence, concave.

To compute a super-gradient, let x be a minimizer, i.e. an eigenvector of $V^T DA + A^T DV - A^T SDA$ corresponding to the smallest eigenvalue. Since this is a point where the minimum is assumed, a super-gradient is given by

$$\frac{\partial \lambda_{\min}}{\partial p} = \left(\frac{\langle 2v_i - s_i a_i \, , x \rangle \langle a_i \, , x \rangle}{\langle a_i \, , v_i \rangle} \right)_{i=1,\dots,m}.$$

□

The previous lemma allows one to solve (5.1) by projected super-gradient ascent as follows: Choose a stepsize sequence t_k and iterate:

1. Initialize with $p_i^0 = 1/m$, $k = 0$
2. Form $V^T DA + A^T DV - A^T SDA$ and compute an eigenvector x corresponding to the minimal eigenvalue.
3. Compute the super-gradient $g_i^k = \frac{\partial \lambda_{\min}}{\partial p}(p^k)$ according to Lemma 5.1.
4. Update $p^{k+1} = \mathrm{proj}_{\Delta_m}(p^k + t_k g^k)$ where proj_{Δ_m} is the projection onto the m-dimensional simplex.

It is worth noting, how this algorithm looks in the special case of $V = A$. There we only want to maximize $\lambda_{\min}(A^T DA)$ and the super-gradient of this at some p^k is just $g^k = \left(\frac{\langle a_i \, , x \rangle^2}{\|a_i\|^2} \right)_{i=1m\dots,m}$. As this is always positive, we can project onto the simplex by a simple rescaling as

$$p^{k+1} = \frac{p^k + t_k g^k}{\|p^k + t_k g^k\|_1}.$$

5.2 Minimizing $\|I - V^T DA\|$

The subgradient of the objective functional in (5.2) is given by the next lemma:

Lemma 5.2 *Let $s_i = \langle a_i, v_i \rangle$. The function $f(p) = \|I - V^T DA\|$ with $D = \mathrm{Diag}(p/s)$ is convex. A subgradient is given by*

$$-\frac{(Aq_1) \odot (Vr_1)}{s} \in \partial f(p)$$

where q_1 and r_1 are left and right singular vectors of $I - V^T DA$ corresponding the largest singular value, $\odot$ denotes component-wise product, and the division is also to be understood component-wise.

Proof The convexity of f follows from the convexity of the norm and the fact that the map $M : \mathbb{R}^m \to \mathbb{R}^{n \times n}$, $p \mapsto -V^T DA$ is linear in p.

Example 1 in [11] shows that the subgradient of the spectral norm is given as follows: If B has a singular value decomposition $B = Q\Sigma R^T$ and the maximal singular value has multiplicity j, then

$$\partial_B \|B\| = \mathrm{conv}\{q_i r_i^T \mid i = 1, \ldots, j\}$$

where q_i and r_i are the ith columns of Q and R, respectively.

By the chain rule for subgradients, we get that

$$\partial_p f(p) = M^* \partial \|I - Mp\|.$$

where the adjoint of M is the map $M^* : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ defined by the property that for all $p \in \mathbb{R}^m$ and all $B \in \mathbb{R}^{n \times n}$ it holds that $\langle p, M^* B \rangle = \langle Mp, B \rangle$. Note that we do not consider M as a matrix (e.g. of size $m \times n^2$) but as a linear map, that we do not vectorize the $n \times n$ matrices, and that we use the inner product $\langle A, B \rangle = \mathrm{trace}(A^T B)$ for square matrices. We calculate (recalling the abuse of notation for Diag operator from the intoduction)

$$\begin{aligned}
\langle p, M^* B \rangle &= \langle Mp, B \rangle \\
&= \mathrm{trace}((Mp)^T B) \\
&= -\mathrm{trace}(V^T \mathrm{Diag}(p/s) AB) \\
&= -\mathrm{trace}(\mathrm{Diag}(p/s) ABV^T) \\
&= \langle p, -\mathrm{Diag}(\mathrm{Diag}(1/s) ABV^T) \rangle,
\end{aligned}$$

and this means that $M^* B \in \mathbb{R}^m$ is given as

$$M^* B = -\mathrm{Diag}\left(\mathrm{Diag}(1/s) ABV^T\right).$$

Plugging in the previous formula we obtain that

$$-\mathrm{Diag}(\mathrm{Diag}(1/s) Aq_1 r_1^T V^T) = -\mathrm{Diag}((Vr_1)^T \mathrm{Diag}(1/s)(Aq_1)) = \frac{(Aq_1) \odot (Vr_1)}{s}$$

is a subgradient of f, which shows the assertion. □

Similar to the previous subsection we can solve (5.2) by projected subgradient descent as follows: Choose a stepsize sequence t_k and iterate:

1. Initialize with $p_i^0 = 1/m$, $k = 0$
2. Compute a pair q, r of left and right singular vectors of $I - V^T DA$ corresponding to the largest singular vector.
3. Compute a subgradient g^k according to Lemma 5.2
4. Update $p^{k+1} = \mathrm{proj}_{\Delta_m}(p^k + t_k g^k)$ where proj_{Δ_m} is the projection onto the m-dimensional simplex.

Both the routines to find optimized probabilities in this and the previous section need spectral decompositions. In Section 5.1 we need an eigenvector corresponding to the smallest eigenvalue of a symmetric matrix and in this section we need a pair of left and right singular vectors corresponding to the largest singular values. These operations are the most costly ones within the method and in fact, in may not be doable in some applications. However, the experiments in Section 6 show that there often exist much "better" probabilties than the widely used $p_i = \|a_i\|_2 / \|A\|_F^2$ (or $p_i = \langle a_i, v_i \rangle / \|A\|_V^2$ in the case of mismatch).

6 Numerical experiments

In this section we report a few numerical experiments that illustrate the results.[2] We start with an illustration of Theorem 2.2, i.e. the consistent and overdetermined case. We used a Gaussian matrix $A \in \mathbb{R}^{500 \times 200}$ (i.e., the entries are independently and normally distributed). To obtain a mismatched adjoint V we simply set all entries of A with magnitude smaller that 0.5 to zero.[3] The unique solution $\hat{x}$ was also generated as a Gaussian vector and as probabilities we used $p_i = \|a_i\|^2 / \|A\|_F^2$. The convergence condition (2.2) is fulfilled and $1 - \lambda \approx 1 - 5.5 \cdot 10^{-4}$ and we also have $\rho(I - V^T DA) \approx 1 - 7.5 \cdot 10^{-4}$. Figure 6.1 shows the error and the residuals for the randomized Kaczmarz method with and without mismatched adjoint. For both the error and the residual, the results for the mismatched adjoint are quite close to the case with exact adjoint (while the residual is slightly smaller for the former, but the error is slightly smaller for the latter). We note that this is not universal: other random instances constructed in the same way show different behavior, although both methods are always quite close to each other.

Our second numerical example treats the inconsistent and overdetermined case. The matrix A and solution $\hat{x}$ and the probabilities are similar to the previous example, but now the right hand side is $b = A\hat{x} + r$ (with Gaussian r). Figure 6.2 shows the result of the RKMA method on this example and also the error bound from Theorem 3.1. As predicted, the error does not go to zero, but levels out at a non-zero level (the same is true for the residual). As in the previous example one sees that the upper bound from Theorem 3.1 is quite loose.

[2] The code to produce the figures in this article is available at `https://github.com/dirloren/rkma`.

[3] We could also add a perturbation to A - the numerical results would be rather similar. Setting entries to zero would be numerically beneficial if A would have many small entries, but this was not be the motivation here.

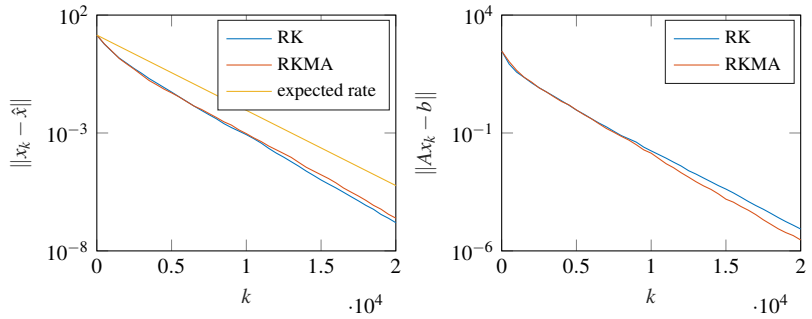


Fig. 6.1 Comparison of the randomized Kaczmarz method with and without mismatched adjoint in the overdetermined and consistent case. Left: Decay of the error and also the asymptotic rate $\rho(I - V^T DA)^k$ from Theorem 2.2. Right: Decay of the residual.

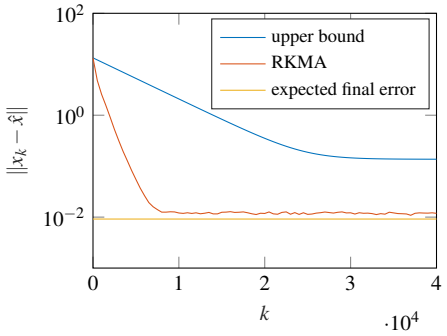


Fig. 6.2 The RKMA method in the inconsistent case. The plot shows the decay of the error, the theoretical upper bound (3.2), and the expected final error $\|\sum_{l=0}^{\infty} M^l V^T Dr\|$ from (3.1) from Theorem 3.1.

Now we illustrate the behavior of RKMA in the underdetermined case. We used $A, V \in \mathbb{R}^{100\times 500}$, again A with Gaussian entries and we obtained V from A by setting the entries of A to zero that have magnitude smaller than 0.3. The solution $\hat{x}$ was constructed as $\hat{x} = V^T c$ for some random vector c and the right hand side was obtained through $b = A\hat{x}$. Hence, generically $\hat{x}$ is not in the range of A^T and the standard randomized Kaczmarz method can not converge to $\hat{x}$. Figure 6.3 shows that the error decays quickly to zero for RKMA but not for the standard randomized Kaczmarz method. The residuals, however, behave similarly for both methods.

For another illustration of the underdetermined case, we consider a toy problem from computerized tomography: For a 50×50 pixel image we generated a CT projection matrix for a parallel beam geometry with 36 equi-spaced angels and 150 rays per angle, which gives a projection matrix of size $5,400 \times 2,500$ (we used the AIRtools package from [5] and the respective MATLAB command is `Afull = paralleltomo(50,0:5:180,150,70)`). For the matrix A for the forward projection we used every third row of the matrix while for V (the backprojection) we used the average of three consecutive rows, thereby employing a simple model for detector bin width in the backprojection operation. Then we eliminated the rows of A and

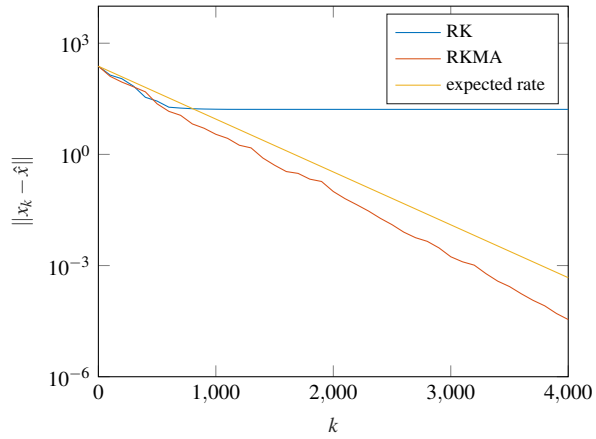


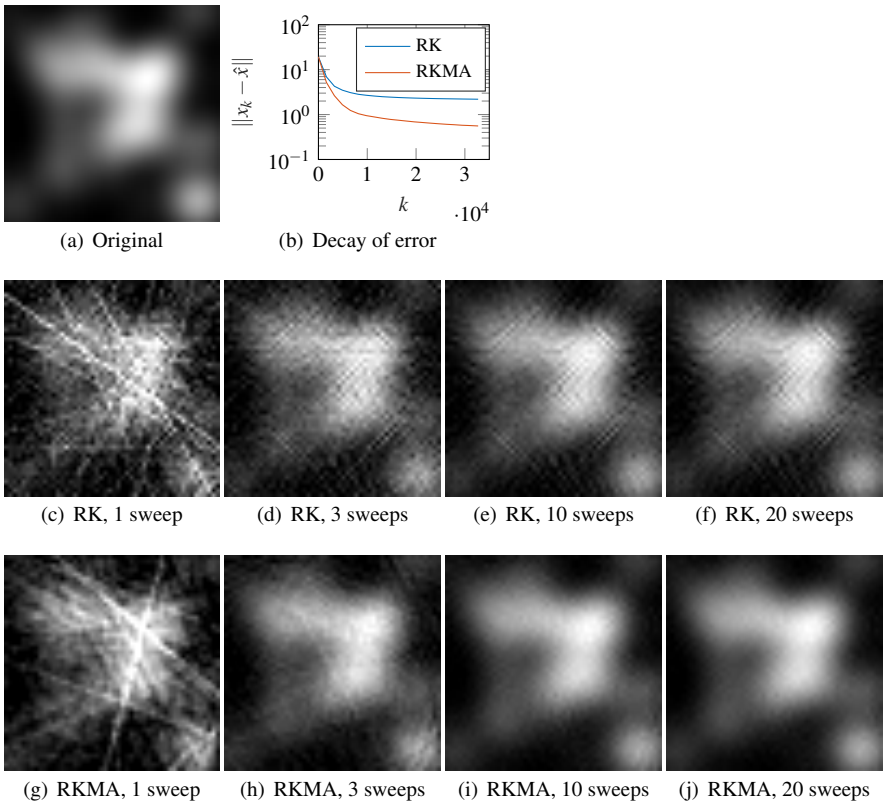
Fig. 6.3 Comparison of the randomized Kaczmarz method with and without mismatched adjoint in the underdetermined and consistent case but with true solution $\hat{x} \in \operatorname{rg} V^T$ and $\hat{x} \notin \operatorname{rg} A^T$. Plot shows the decay of the error.

V which correspond to zero-rows in A which leaves us with two matrices of size $1,636 \times 2,500$. Then we generated a smooth image by

```
im = phantomgallery('ppower',N,0.3,1.3,155432);
im = imfilter(im,fspecial('gaussian',16,4));
im = im/max(max(im));
x = im(:);
```

and generated the data by `b = A*x`. We reconstructed x by RKMA and RK (with the probabilities $p_i = \|a_i\|_2^2 / \|A\|_F^2$ from Remark 2.4). Figure 6.4 shows the reconstructions after a quite small number of sweeps (one sweep corresponds to m steps of the methods, where m is is the number of rows). One sees that using a mismatched adjoint is beneficial in this setting: First, the iteration converges to a limit which is closer to the original image (which is due to the fact that this is closer to the range of V^T than to that of A^T since the image is smoother than the elements in the range of A^T). Moreover, the initial iterates are better. As expected, the reconstruction with A^T and A suffers from Moire patterns. Using V^T as adjoint avoids these artifacts as the range of V^T contains smoother functions, in some sense. Finally, we note that applying RK using V for both the forward and back-projection does also converge, but leads to an even worse reconstruction than using RK with A.

Finally, we illustrate that the optimization of the probabilities according to Section 5 does indeed improve the practical performance. We used $A, V \in \mathbb{R}^{300 \times 100}$ where A is a random matix with Gaussian entries where the ith row has been scaled with the factor $2/(\sqrt{i}+2)$ and V has been obtained from A by setting 5% randomly chosen entries of A to zero. We calculated optimized probabilities by the methods from Sections 5.1 and 5.2, respectively (initialized with uniform probabilities, applied 600 iterations of the method to maximize λ and 200 iterations to minimize $\|I - V^T DA\|$, in both cases with decaying stepsizes $t \approx 1/k$). We applied RKMA with these optimized probabilities, uniform probabilities, and $p_i \propto \langle a_i, v_i \rangle$. Figure 6.5 shows that the



(a) Original (b) Decay of error

(c) RK, 1 sweep (d) RK, 3 sweeps (e) RK, 10 sweeps (f) RK, 20 sweeps

(g) RKMA, 1 sweep (h) RKMA, 3 sweeps (i) RKMA, 10 sweeps (j) RKMA, 20 sweeps

Fig. 6.4 Reconstruction for a toy CT example.

optimized probabilites indeed outperform the uniform choice and the choice proportional to $\langle a_i, v_i \rangle$. Table 6.1 shows the respective quantities for the different probabilities. Although both approaches optimize different quantities and neither optimizes the asymptotic convergence rate, both probabilities are rather similar in practice, and, as shown in Figure 6.5 on the right, the probabilities for the different optimization problems are quite similar.

Table 6.1 Quantities describing the convergence of RKMA for different probabilities.

	unif	row	$\max \lambda$	$\min \|I - V^T DA\|$
$1 - \lambda$	0.998588	0.999079	0.997820	0.998311
$\rho(I - V^T DA)$	0.997908	0.998352	0.997540	0.997327
$\|I - V^T DA\|$	0.998029	0.998485	0.997752	0.997439

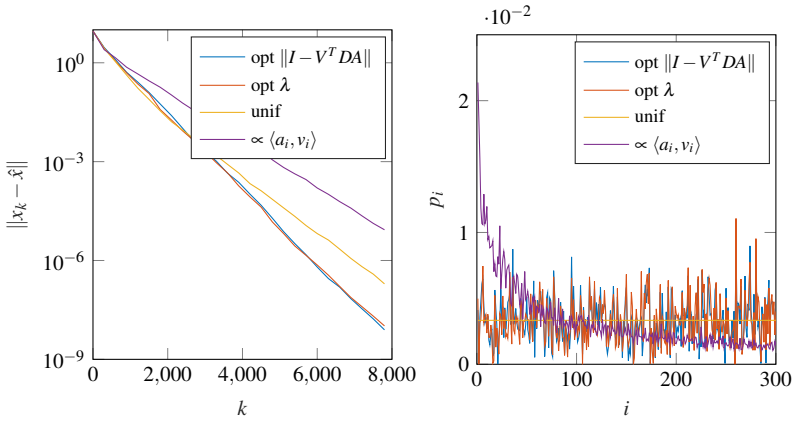


Fig. 6.5 A sample run of RKMA on a consistent system with different probabilities: "unif" refers to the uniform probabilities $p_i = 1/m$, "$\propto \langle a_i, v_i \rangle$" uses $p_i = \langle a_i, v_i \rangle / \|A\|_V^2$ (cf. Remark 2.4) and "opt $\|I - V^T DA\|$" and "opt λ" refer to probabilities obtained by the methods from Section 5.

7 Conclusion

We derived several results on the convergence of the randomized Kaczmarz method with mismatched adjoint and could show that the method converges linearly when the mismatch is not too large. The results are a little bit more complicated compared to the case of no mismatch due to the asymmetry of the matrix $I - V^T DA$. In particular, estimates for the norm of the expected error and the expectation of the norm of the error are different in this case. We were also able to characterize the asymptotic convergence rate of RKMA and numerical experiments indicate that this estimate of the rate is indeed quite sharp. In the underdetermined case one may even take advantage of the use of a mismatched adjoint to drive the randomized Kaczmarz method to a solution in the subspace $\operatorname{rg} V^T$. This last point may be important for algebraic reconstruction techniques in computerized tomography where mismatched projector pairs are often employed. Using the conditions derived here, a thorough study of commonly used mismatched projector pairs could be performed to determine what pairs have guaranteed asymptotic convergence properties. We did not investigate other uses of mismatched adjoints, but it may be that certain control problems could benefit from some freedom in the adjoint when the adjoint corresponds to the solution of an adjoint state equation.

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