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Existence Tests for uncertain functions with parameters

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Introduction

A mobile robot is moving on an horizontal plane :

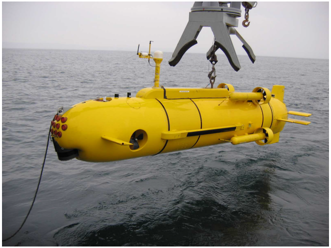


Figure : Redermor underwater robot.

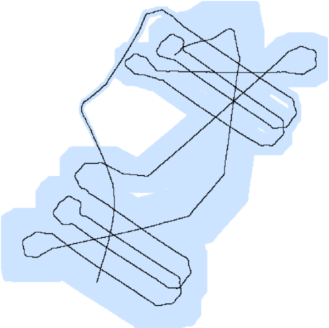


Figure : Tube enclosing the trajectory **p** of the robot.

Detecting Loops is an important topic in SLAM !

Introduction

Algorithm LOOP [ADJ] explore the *t-plane* to find inner/outer approximation of the set :

$$\mathbb{T}^* = \left\{ (t_1, t_2) \in [0, t_{\max}]^2, \mathbf{p}(t) = \int_0^t \mathbf{v}(\tau)\, d\tau \text{ and } t_1 < t_2 \right\} \quad (1)$$

The algorithm return *t-boxes* classified in $\mathbb{T}^{out}, \mathbb{T}^{in}$ and $\mathbb{T}^{?}$.

$$\mathbb{T}^{in} \subset \mathbb{T} \subset \left(\mathbb{T}^{in} \cup \mathbb{T}^{?} \right) \quad (2)$$

with $\mathbb{T}$ enclosing $\mathbb{T}^*$

Introduction

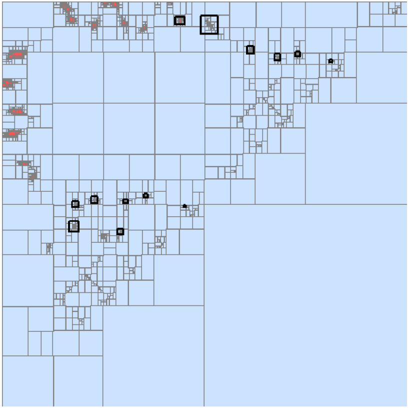


Figure : *t-plane*

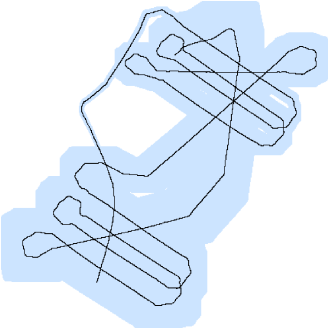


Figure : Tube enclosing the trajectory **p** of the robot.

Newton test for Existence And Uniqueness

Consider a smooth function $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $[\mathbf{x}] \in \mathbb{R}^n$.
Denote by $\mathbf{J_f}$ is Jacobian Matrix.

Definition (Newton test, Moore [MKC09])

$$\mathcal{N}\left(\mathbf{f}, [\mathbf{J_f}], [\mathbf{x}]\right) = \widehat{\mathbf{x}} - [\mathbf{J_f}]^{-1}\left([\mathbf{x}]\right).\mathbf{f}\left(\widehat{\mathbf{x}}\right) \tag{1}$$

If $\mathcal{N}([\mathbf{x}]) \subset [\mathbf{x}]$ then $[\mathbf{x}]$ contains a unique zero $\mathbf{x}^*$ of f.It is also in $\mathcal{N}([\mathbf{x}])$.

The Newton test applied in LOOP algorithm prove the unicity and existence of 14 loops over 28. And we wants more !

Krawczyc test

Consider a smooth function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ and $[\mathbf{x}] \in \mathbb{R}^n$.

Theorem (Krawczyk test, Moore [MKC09])

Let Y be a nonsingular matrix approximating $J_f(center([\mathbf{x}]))^-1$. Let $\widehat{\mathbf{x}} \in [x]$ a real vector.

$$K([\mathbf{x}]) = \widehat{\mathbf{x}} - Y.f(\widehat{\mathbf{x}}) + \{I - Y.J_f([\mathbf{x}])\}.([\mathbf{x}] - \widehat{\mathbf{x}}) \tag{2}$$

If $K([\mathbf{x}]) \subseteq [\mathbf{x}]$ then $[\mathbf{x}]$ contains a zero $\mathbf{x}^*$ of f.It is also in $K([\mathbf{x}])$.

Miranda test for Existence

Consider a smooth function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$

Theorem (Miranda[Mir40])

Define : $[\mathbf{x}]^{i\pm} = ([x_1], ..., [x_{i-1}], inf\,/sup([x_i]), [x_{i+1}], ..., [x_n])$.

If

$$\left.\begin{array}{l} f_i(\mathbf{x}) \geqslant 0 \forall x \in [\mathbf{x}]^{i-}, \\ f_i(\mathbf{x}) \leqslant 0 \forall x \in [\mathbf{x}]^{i+}, \end{array}\right\}, i = 1, ..., n \qquad (3)$$

then $[x]$ *contains a zero* x^* *of f.*

Miranda test with bissection

When the functions involved are not well conditionned, Miranda can't prove anything.

midpoint inverse preconditionning

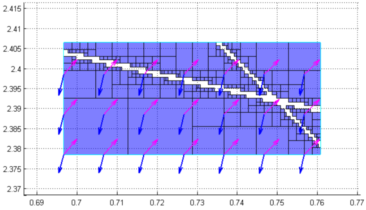


Figure : Can't apply Miranda on facets

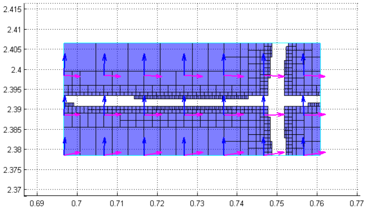


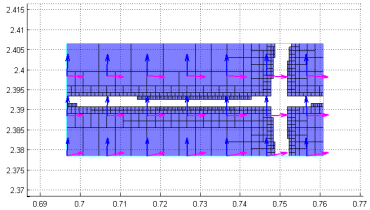
Figure : Could now apply Miranda

Miranda test with bissection

Even if we're well conditionned, Miranda can fail as in the example at the left where :

$$f_2([x_1], inf([x_2])) \not\leq 0$$

That happens cause of small variations of ∇_{f2} (in magenta) and $width([x])$.
If we can't change ∇_{f2}, we can work on $[x]$.
The Idea is to bissect $[x]$ in Miranda test.



On A Robot Trajectory

A mobile robot is moving on a trajectory defined by :

$$f(t,[p]) = \begin{pmatrix} x(t,[p]) \\ y(t,[p]) \end{pmatrix} = \begin{pmatrix} \sin(t + p_1 * t).cos(2.t + p_2 * t) \\ sin(t^2 + p_3 * t).cos(t + p_4 * t) \end{pmatrix}$$

with
$p = ([0.0050, 0.015][0.105, 0.115][-0.095, -0.085][0.205, 0.215])^T$
Now assume LOOP algorithm on the velocity tubes of this trajectory return *t-boxes* $([\mathbf{t_1}], [\mathbf{t_2}]) \in [0, t_{max}]^2$ that satisfies :

$$g(t_1, t_2) = f(t_2) - f(t_1) = 0 \tag{4}$$

and

$$0 < t1 < t2$$

We want to prove the existence and uniqueness of $g(t_1, t_2) = 0$ with $t_1 \in [t_1], t_2 \in [t_2]$.

On A Robot Trajectory

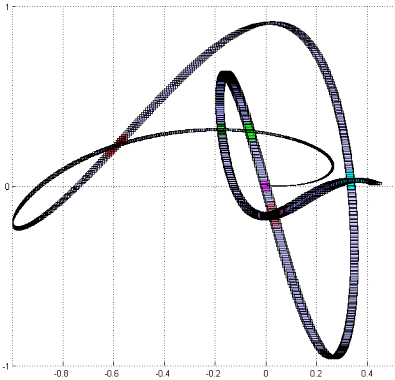


Figure : Trajectory of the mobile robot.

On A Robot Trajectory

We apply existence tests presented before to our trajectory and the following table answer to the question "Did the test guarantee the existence of $f(x) = 0$ for the i_{th}intersection ?"

intersection	$\mathcal{N}([\mathbf{x}])$	$K([\mathbf{t}])$	Miranda	Miranda B
1	no	no	no	no
2	no	no	no	no
3	yes	yes	no	yes
4	no	no	no	yes
5	no	no	no	yes
6	no	no	no	yes

Discussion

- More and more test for Miranda-Bissection method.

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- The Question is : How Bissection Could Improve Existence Tests ?

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- More and more test for Miranda-Bissection method.
- The Question is : How Bissection Could Improve Existence Tests ?
- What about Uniqueness ?

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