Semigroup approach for validating solutions to semilinear parabolic PDEs

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Solving IVP of semilinear parabolic PDEs

$$\begin{cases} u_t = (\lambda_0 + \lambda_1 \Delta + \lambda_2 \Delta^2) u + N(u, \nabla u, \Delta u), & t > 0, \quad x \in \Omega \\ u(0, x) = u_0(x), & x \in \Omega. \end{cases}$$

- $u \equiv u(t, x) \in \mathbb{R}$ or $\mathbb{C}$: unknown, $u_0(x)$: initial data
- $\Omega = \mathbb{T}^d$ ($d = 1, 2, 3$) or polygonal: Periodic b.c. / cosine symmetry (Neumann b.c.) / sine symmetry (Dirichlet b.c.)
- λ_0, λ_1, $\lambda_2 \in \mathbb{R}$ are chosen so that PDE is *parabolic*
- The derivative in N is less than that in the linear part

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Rigorous Integrators

[Zgliczyński '04,'10] [Cyranka '14,'15] [Cyranka-Zgliczyński '15] [Wilczak-Zgliczyński '20] [Kalita-Zgliczyński '21] [Arioli-Koch '10] [Mizuguchi-T.-Kubo-Oishi '17] [T.-L-Jaquette-Okamoto '22] [Duchesne-Lessard-T. '25] [van den Berg-Breden-Sheombarsing '24] [Cyranka-Lessard '22] [Cadiot-Lessard '25] [Hashimoto-Kinoshita-Nakao '20] etc.

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$\boxed{\text{Contraction mapping theorem}}$ (Newton–Kantrovich type th.)

Let $\mathcal{X}$ be a Banach space, $\bar{u} \in \mathcal{X}$, and $T : \mathcal{X} \to \mathcal{X}$ a continuously differentiable map. Assume that there exists a constant Y and a non-decreasing map $Z : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} \|T(\bar{u}) - \bar{u}\| &\le Y \\ \|DT(u)\| &\le Z(\|u - \bar{u}\|), \quad \forall u \in \mathcal{X}. \end{aligned}$$

If there exists $r > 0$ such that

$$\begin{aligned} Y + \int_0^r Z(s)ds &\le r \\ Z(r) &< 1, \end{aligned}$$

then T has a unique fixed point $\tilde{u}$ such that $\|\tilde{u} - \bar{u}\| \le r$ holds.

Sketch of proof

For $\forall u \in B_r(\bar{u}) \stackrel{\text{def}}{=} \{u \in \mathcal{X} : \|u - \bar{u}\| \le r\}$,

$$\begin{aligned}\|T(u) - \bar{u}\| &\le \|T(u) - T(\bar{u})\| + \|T(\bar{u}) - \bar{u}\| \\ &\le \int_0^r Z(s)ds + Y \le r,\end{aligned}$$

then $T(u) \in B_r(\bar{u})$ holds. Furthermore, for $\forall u_1,\ u_2 \in B_r(\bar{u})$,

$$\begin{aligned}\|T(u_1) - T(u_2)\| &\le \sup_{\theta\in[0,1]} \|DT(\theta u_1 + (1-\theta)u_2)\|\|u_1 - u_2\| \\ &\le Z(r)\|u_1 - u_2\| < \|u_1 - u_2\|.\end{aligned}$$

This yields $\exists! \tilde{u} \in B_r(\bar{u})$ such that $\tilde{u} = T(\tilde{u})$ based on Banach's fixed-point theorem.

Rigorous integrator for IVPs

- Which function space $\mathcal{X}$ we choose? $\Longrightarrow$
  Fully spectral approach [Cadiot-Lessard '25]:

$$\mathcal{X} = \left\{ (U_{k,n})_{(k,n)\in\mathbb{Z}^m\times\mathbb{N}_0} \in \ell^1 : \|U\|_\omega \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^m} \sum_{n\in\mathbb{N}_0} |U_{k,n}|\omega_{k,n} \right\}.$$

  Semigroup approach [van den Berg-Breden-Sheombarsing '24]:

$$\mathcal{X} = \left\{ (u_k(t))_{k\in\mathbb{Z}^d} \in \ell^1(C^0([0,h])) : \|u\| \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}^d} \sup_{t\in J} |u_k(t)|\,\omega_k \right\}.$$

- How to define T ? $\Longrightarrow$ Newton-like operator

IVP of (finite-dimansional) ODEs

$$\begin{cases} \dot{u} = f(u,t) \\ u(0) = u_0 \end{cases} \quad \Rightarrow \quad T(u)(t) = U(t,s)u_0 + \int_0^t U(t,s)G(u(s))ds.$$

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Autonomous semigroup approach

$$U(t,s) = e^{A(t-s)}, \quad G(u(s)) = f(u,s) - Au(s), \quad A \sim \overline{Df(\bar{u})}.$$

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Non-autonomous semigroup approach

$U(t,s)$: Fundamental sol. matrix, $\quad G(u(s)) = f(u,s) - Df(\bar{u}(s),s)u(s).$

IVP of (infinite-dimansional) ODEs

$$\begin{cases} \dot{u} = \mathcal{L}u + \mathcal{Q}\mathcal{N}(u) \\ u(0) = u_0 \end{cases} \Rightarrow \quad T(u)(t) = U(t,s)u_0 + \int_0^t U(t,s)G(u(s))ds.$$

Autonomous semigroup approach*

$$U(t,s) = e^{\mathcal{A}(t-s)}, \quad G(u(s)) = \mathcal{L} + \mathcal{Q}\mathcal{N}(u,s) - \mathcal{A}u(s), \quad \mathcal{A} \sim \mathcal{L} + \overline{\mathcal{Q}D\mathcal{N}(\bar{u})}.$$

Non-autonomous semigroup approach†

$U(t,s)$: Evolution operator, $\quad G(u(s)) = \mathcal{Q}(\mathcal{N}(u(s)) - D\mathcal{N}(\bar{u}(s))u(s)).$

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*J. B. van den Berg, M. Breden, R. Sheombarsing, Numerische Mathematik (2024) 156:1219–1287.

†G. W. Duchesne, J.-P. Lessard, A. T., Journal of Scientific Computing (2025) 102:62.

Non-autonomous semigroup approach

Control the evolution operator $U(t,s)$ by obtaining the bounds $\boldsymbol{W^{(J,0)}}$ and $\boldsymbol{V^{(J,0)}}$ such that

$$\|U(\cdot,0)\|_{B(\ell^1_\omega,\mathcal{X})} \le \boldsymbol{W^{(J,0)}}, \qquad \left\| \int_0 U(\cdot,s)ds \right\|_{B(\mathcal{X})} \le \boldsymbol{V^{(J,0)}}.$$

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This yields

$$\|T(\bar{u})-\bar{u}\| \le \boldsymbol{W}^{(\boldsymbol{J},\mathbf{0})}\|u_0-\bar{u}(0)\|_\omega + \boldsymbol{V}^{(\boldsymbol{J},\mathbf{0})}\|\dot{\bar{u}}-\mathcal{L}\bar{u}-\mathcal{Q}\mathcal{N}(\bar{u})\| \overset{\text{def}}{=} Y$$

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and for any $u \in B_r(\bar{u})$

$$\|DT(u)\|_{B(\mathcal{X})} \le \boldsymbol{V}^{(\boldsymbol{J},\boldsymbol{0})}\tilde{Z}(r) \stackrel{\text{def}}{=} Z(r),$$

where $\tilde{Z}$ is a non-decreasing functions $\tilde{Z}:[0,\infty)\to[0,\infty)$ s.t.

$$\|DG(u)\|_{B(\mathcal{X})} \le \tilde{Z}(r), \quad \forall u \in B_r(\bar{u}).$$

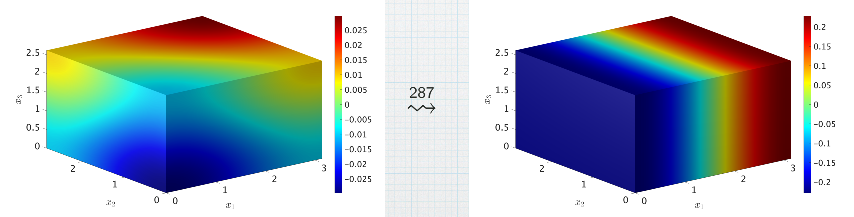
Swift-Hohenberg equation[‡] (3D)

$$u_t = \sigma u - (1+\Delta)^2 u - u^3, \quad \sigma \in \mathbb{R}$$

with the Neumann b.c. on $[0,\pi] \times [0,\pi/1.1] \times [0,\pi/1.2]$.

A trajectory to stable equilibrium (stripe pattern)

($\sigma = 0.04$, Fourier trunc. $\boldsymbol{N} = 4$, step size $h = 0.25$, 287 steps):



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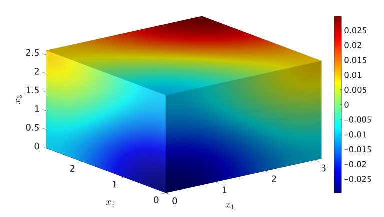
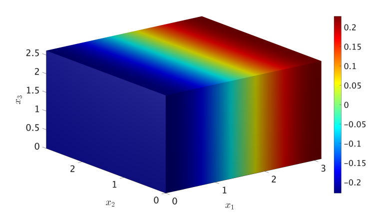
A trajectory to stable equilibrium (stripe pattern)

($\sigma = 0.04$, Fourier trunc. $\boldsymbol{N} = 6$, step size $h \leq 1.0$, 73 steps):



$\overset{73}{\rightsquigarrow}$



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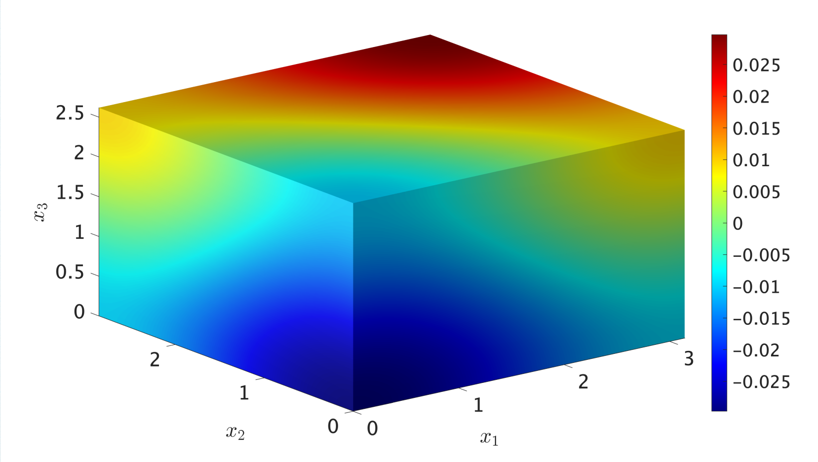
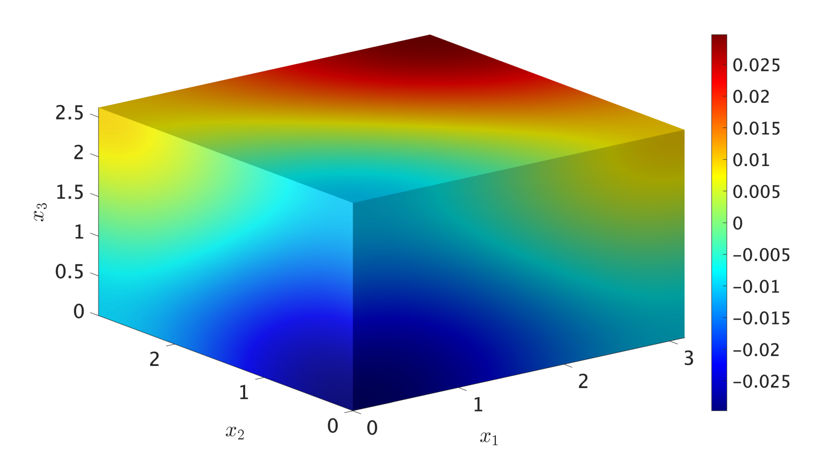


Time evolution of 3D-SH is rigorously included by

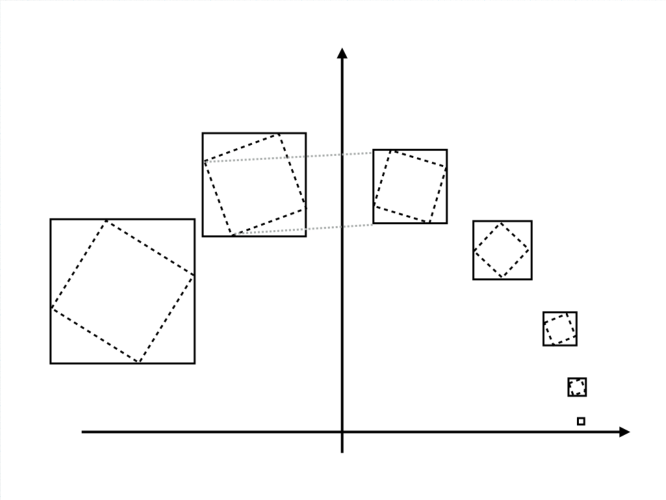
$$\sup_{t\in[0,71.75]} \|u(t)-\bar{u}(t)\| \le 4.7241\cdot 10^{-6}$$



Time evolution of 3D-SH is rigorously included by

$$\sup_{t\in[0,72]} \|u(t)-\bar{u}(t)\| \le 5.8117\cdot 10^{-10}$$

Against the wrapping effect in PDEs



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- Time stepping is not good

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- Validate solution over multi-steps
- Integrating only one (long) interval

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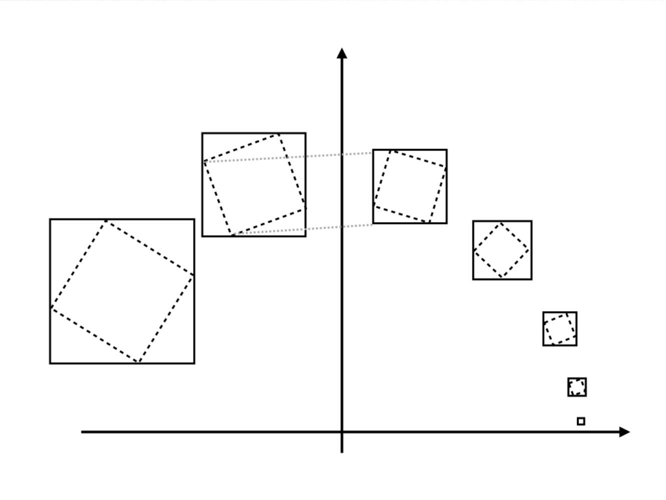

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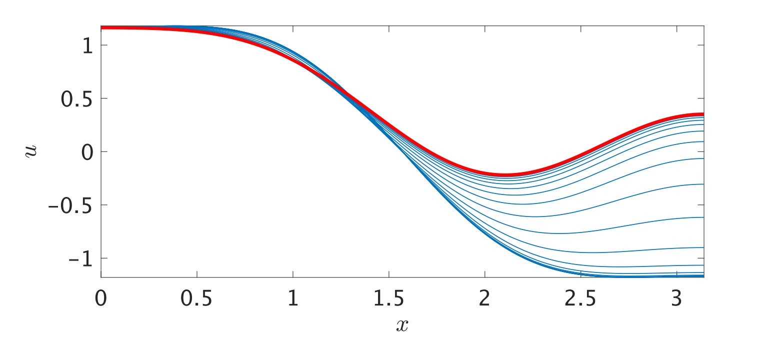
Improved result by multi-step scheme



1D Swift-Hohenberg eq. converging to an equilibrium (Morse indx 1)

Solve the IVP for $t \in [0, 10]$ with the step size 0.1. Initial profile (red) transforms to the one near unstable equilibrium. This is a good example to control the error propagation.

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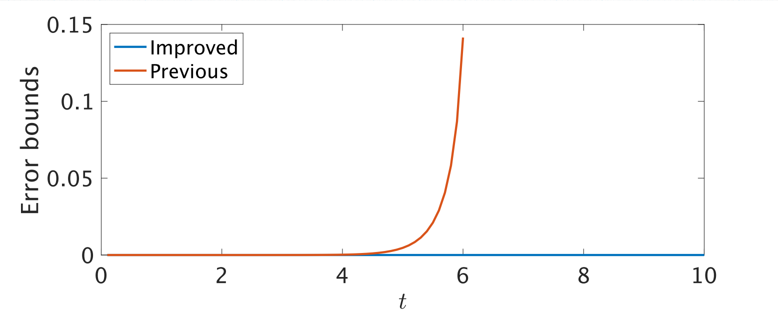

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Previous integrator [Duchesne-Lessard-T. '25] fails to solve IVP around $t = 6$ due to the error propagation ($\approx$ 97 mins). Our improved multi-step scheme succeeds to solve untile $t = 10$ with 10^{-9} accuracy ($\approx$ 9 mins).

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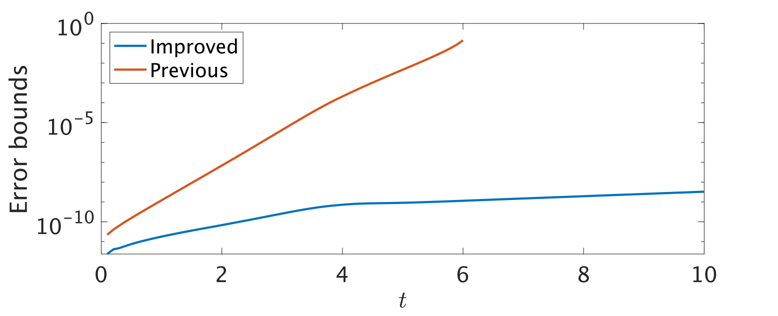

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Infinite-dimensional ODEs on ℓ^1_ν

$$\dot{u} = \mathcal{L}u + \mathcal{N}(u), \quad u(0) = u_0, \quad t \in [0, \tau].$$

- $(\mathcal{L}\phi)_{\boldsymbol{k}} \stackrel{\text{def}}{=} \mu_{\boldsymbol{k}}\phi_{\boldsymbol{k}} : D(\mathcal{L}) \subset \ell^1_\nu \to \ell^1_\nu$, $D(\mathcal{L}) \stackrel{\text{def}}{=} \{\phi : \mathcal{L}\phi \in \ell^1_\nu\}$
- $(\mathcal{N}(\phi))_{\boldsymbol{k}} \stackrel{\text{def}}{=} \mathcal{N}_{\boldsymbol{k}}(\phi) : \ell^1_\nu \to \ell^1_\nu$, Fréchet differentiable, involving discrete convolutions, $\mathcal{N}(0) = 0$ & $D\mathcal{N}(0) = 0$

Banach space for the sequence:

$$\ell^1_\nu \stackrel{\text{def}}{=} \left\{ a = (a_{\boldsymbol{k}})_{\boldsymbol{k}\geq 0} : a_{\boldsymbol{k}} \in \mathbb{R},\ \|a\|_\nu \stackrel{\text{def}}{=} \sum_{\boldsymbol{k}\geq 0} |a_{\boldsymbol{k}}|\nu^{|\boldsymbol{k}|} < +\infty \right\},\ \nu \geq 1.$$

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Ensure that ℓ^1_ν is a Banach algebra under the discrete convolution.

Multi-step scheme – setting up –

Time steps: $J_m = [\tau_{m-1}, \tau_m]$ $(m = 1, 2, \ldots, M)$, where

$$0 = \tau_0 < \tau_1 < \cdots < \tau_M = \tau, \quad h_m \stackrel{\text{def}}{=} \tau_m - \tau_{m-1}.$$

Solution: $u_m(t) \stackrel{\text{def}}{=} u(t)$ $(t \in J_m)$, its approximation: $\bar{u}(t) = \bar{u}_m(t)$.

Set J_m dependent function space

$$X_m \stackrel{\text{def}}{=} \ell^1_\nu(C^0(J_m)) = \left\{u : \|u\|_{X_m} \stackrel{\text{def}}{=} \left\| \|u\|_{C^0(J_m)} \right\|_\nu < +\infty\right\}$$

and the neighborhood of the approximate solution on each time step

$$B_{r_m}(\bar{u}) \stackrel{\text{def}}{=} \{u \in X_m : \|u - \bar{u}\|_{X_m} \le r_m\}.$$

Validate the solution in a Banach space:

$$\prod_{m=1}^{M} B_{r_m}(\bar{u}) \subset \prod_{m=1}^{M} X_m \stackrel{\text{def}}{=} \boldsymbol{X} \left(\subset \prod_{m=1}^{M} C(J_m, \ell^1_\nu) \right).$$

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$\boxed{\text{The fixed-point form}}$ For $t \in J_m$, define

$$\left(\tilde{T}_m(u)\right)(t) \stackrel{\text{def}}{=} U_m(t, \tau_{m-1})\sigma_{m-1}(u) + \int_{\tau_{m-1}}^{t} U_m(t, s) G_m(u(s))ds.$$

$U_m(t, s)$ denotes the evolution operator generated by the *non-autonomous* differential operator $\mathcal{L} + D\mathcal{N}(\bar{u}_m(t))$ and

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$$\sigma_m(u) \stackrel{\text{def}}{=} U_m(\tau_m,\tau_{m-1})\sigma_{m-1}(u) + \int_{\tau_{m-1}}^{\tau_m} U_m(\tau_m,s)G_m(u(s))ds$$

and $\sigma_0(u) = u_0$. Note that $\sigma_m(u) = \sigma_m\left(\tilde{T}_m(u)\right)$.

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and $\sigma_0(u) = u_0$. Note that $\sigma_m(u) = \sigma_m\left(\tilde{T}_m(u)\right)$. Furthermore,

$$G_m(u) \stackrel{\text{def}}{=} \mathcal{N}(u) - D\mathcal{N}(\bar{u}_m)u,$$

where $DG_m(u) = D\mathcal{N}(u) - D\mathcal{N}(\bar{u}_m)$ and $DG_m(\bar{u}) = 0$ hold.

The explicit form of $\tilde{T}_m$:

$$
\begin{aligned}
&\left(\tilde{T}_m(u)\right)(t) \\
&= U_m(t,\tau_{m-1})\sigma_{m-1}(u) + \int_{\tau_{m-1}}^{t} U_m(t,s)G_m(u(s))ds \\
&= U_m(t,\tau_{m-1}) \left(U_{m-1}(\tau_{m-1},\tau_{m-2})\sigma_{m-2}(u) + \int_{\tau_{m-2}}^{\tau_{m-1}} U_{m-1}(\tau_{m-1},s)G_{m-1}(u(s))ds \right) \\
&\qquad + \int_{\tau_{m-1}}^{t} U_m(t,s)G_m(u(s))ds \\
&= \dots \\
&= U_m(t,\tau_{m-1}) \left(\Phi_{m-1,0}u_0 + \sum_{j=1}^{m-1} \Phi_{m-1,j} \int_{\tau_{j-1}}^{\tau_j} U_j(\tau_j,s)G_j(u(s))ds \right) \\
&\qquad + \int_{\tau_{m-1}}^{t} U_m(t,s)G_m(u(s))ds.
\end{aligned}
$$

The explicit form of $\tilde{T}_m$:

$$\left(\tilde{T}_m(u)\right)(t) = U_m(t,\tau_{m-1})\left(\Phi_{m-1,0}u_0 + \sum_{j=1}^{m-1}\Phi_{m-1,j}\int_{\tau_{j-1}}^{\tau_j} U_j(\tau_j,s)G_j(u(s))ds\right)$$

$$+\int_{\tau_{m-1}}^{t} U_m(t,s)G_m(u(s))ds,$$

where, $\Phi_{m-1,j}$ for $j=0,1,\dots,m-1$ is defined by

$$\begin{aligned}\Phi_{m-1,j} &\overset{\text{def}}{=} U_{m-1}(\tau_{m-1},\tau_{m-2})U_{m-2}(\tau_{m-2},\tau_{m-3})\cdots U_{j+1}(\tau_{j+1},\tau_j)\\ \Phi_{j,j} &\overset{\text{def}}{=} \mathrm{Id}.\end{aligned}$$

Find the fixed-point of the map:

$$\tilde{\boldsymbol{T}}(u) \overset{\text{def}}{=} \left(\tilde{T}_m(u)\right)_{m=1}^{M}, \quad \tilde{\boldsymbol{T}}:\boldsymbol{X}\to\boldsymbol{X}\left(=\prod_{m=1}^{M}X_m\right).$$

The explicit form of $\tilde{T}_m$:

$$\left(\tilde{T}_m(u)\right)(t) = U_m(t, \tau_{m-1}) \left(\Phi_{m-1,0} u_0 + \sum_{j=1}^{m-1} \Phi_{m-1,j} \int_{\tau_{j-1}}^{\tau_j} U_j(\tau_j, s) G_j(u(s)) ds \right)$$

$$+ \int_{\tau_{m-1}}^{t} U_m(t, s) G_m(u(s)) ds,$$

where, $\Phi_{m-1,j}$ for $j = 0, 1, \ldots, m-1$ is defined by

$$\Phi_{m-1,j} \overset{\text{def}}{=} U_{m-1}(\tau_{m-1}, \tau_{m-2}) U_{m-2}(\tau_{m-2}, \tau_{m-3}) \cdots U_{j+1}(\tau_{j+1}, \tau_j)$$

$$\Phi_{j,j} \overset{\text{def}}{=} \mathrm{Id}.$$

Find the fixed-point of the map:

$$\tilde{\boldsymbol{T}}(u) \overset{\text{def}}{=} \left(\tilde{T}_m(u)\right)_{m=1}^{M}, \quad \tilde{\boldsymbol{T}} : \boldsymbol{X} \to \boldsymbol{X} \left(= \prod_{m=1}^{M} X_m \right).$$

$\boxed{\text{Newton–Kantrovich type th.}}$

Let $\bar{u} \in \boldsymbol{X}$ and assume there exist constants Y_m and non-decreasing maps $Z_m^i : [0, \infty) \to [0, \infty)$ such that

$$\begin{aligned} \|\tilde{T}_m(\bar{u}) - \bar{u}\|_{X_m} &\leq Y_m \\ \|D\tilde{T}_m(u)\|_{B(X_i, X_m)} &\leq Z_m^i(r_i), \quad \forall u \in B_{r_i}(\bar{u}) \end{aligned}$$

for $m = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, m$. If there exists $(r_m)_{m=1}^M > 0$ such that

$$Y_m + \sum_{i=1}^m \int_0^{r_i} Z_m^i(s) ds \leq r_m, \qquad \sum_{i=1}^m Z_m^i(r_i) < 1,$$

then $\tilde{\boldsymbol{T}}$ has a unique fixed point $\tilde{u} \in \prod_{m=1}^M B_{r_m}(\bar{u}) \subset \boldsymbol{X}$.

$\boldsymbol{W}^{(J_m,\tau_j)}$ & $\boldsymbol{V}^{(J_m,\tau_j)}$ bounds

Before getting Y_m and Z_m, we assume that there exist the following positive constants for $j = 0, 1, \ldots, m-1$:

$$
\begin{aligned}
\left\|U_m(\cdot,\tau_{m-1})\Phi_{m-1,j}\right\|_{B(\ell^1_\nu,X_m)} &\le \boldsymbol{W}^{(J_m,\tau_j)} \\
\left\|U_m(\cdot,\tau_{m-1})\Phi_{m-1,j+1}\int_{\tau_j}^{\tau_{j+1}} U_{j+1}(\tau_{j+1},s)ds\right\|_{B(X_{j+1},X_m)} &\le \boldsymbol{V}^{(J_m,\tau_j)} \\
&\qquad (j \neq m-1) \\
\left\|\int_{\tau_{m-1}} U_m(\cdot,s)ds\right\|_{B(X_{m-1},X_m)} &\le \boldsymbol{V}^{(J_m,\tau_{m-1})}.
\end{aligned}
$$

These bounds are obtained by solving the linearized IVP along the $\bar{u}$.

Y_m bound

Using $\boldsymbol{W}^{(\boldsymbol{J_m},\boldsymbol{\tau_j})}$ and $\boldsymbol{V}^{(\boldsymbol{J_m},\boldsymbol{\tau_j})}$, the Y_m bound:

$$\|\bar{u}_m - \tilde{T}_m(\bar{u})\|_{X_m} \leq \sum_{j=0}^{m-1} \left(\boldsymbol{W}^{(\boldsymbol{J_m},\boldsymbol{\tau_j})} Y_2^{j+1} + \boldsymbol{V}^{(\boldsymbol{J_m},\boldsymbol{\tau_j})} Y_1^{j+1}\right) \stackrel{\text{def}}{=} Y_m.$$

Here, we denoted

$$y_1^j(s) \stackrel{\text{def}}{=} \frac{d}{ds}\bar{u}_j(s) - \mathcal{L}\bar{u}_j(s) - \mathcal{N}(\bar{u}_j(s)), \quad y_2^j \stackrel{\text{def}}{=} \sigma_{j-1}(\bar{u}_j - \bar{u}_{j-1}) \quad (j = 1, 2, \dots, m)$$

with computable bounds $\left\|y_1^j\right\|_{X_j} \leq Y_1^j$ and $\left\|y_2^j\right\|_\nu \leq Y_2^j$.

Z_m^i bound

Z_m^i bounds for $i = 1, 2, \ldots, m$:

$$\|D\tilde{T}_m(u)\|_{B(X_i, X_m)} = \left\| U_m(\cdot, \tau_{m-1}) \Phi_{m-1,i} \int_{J_i} U_i(\tau_i, s) DG_i(u(s)) ds \right\|_{B(X_i, X_m)}$$

$$\leq \boldsymbol{V^{(J_m, \tau_{i-1})}} \tilde{Z}_i(r_i) \stackrel{\text{def}}{=} Z_m^i(r_i).$$

Here, we assume that $\exists \tilde{Z}_m : [0, \infty) \to [0, \infty)$ s.t.

$$\|DG_m(u)\|_{B(X_m)} \leq \tilde{Z}_m(r_m), \quad \forall u \in B_{r_m}(\bar{u}), \quad m = 1, 2, \ldots, M.$$

Summary

- Semigroup approach: Newton-Kantorovich (contraction mapping) argument using the evolution operator
- Multi-step scheme to control the wrapping effect
- This approach does not require huge inverse matrix

Future directions

- Saddle-to-saddle connection in PDEs
- Navier–Stokes equations
  Global existence from some initial data / Global dynamics of axisymmetric NSEs

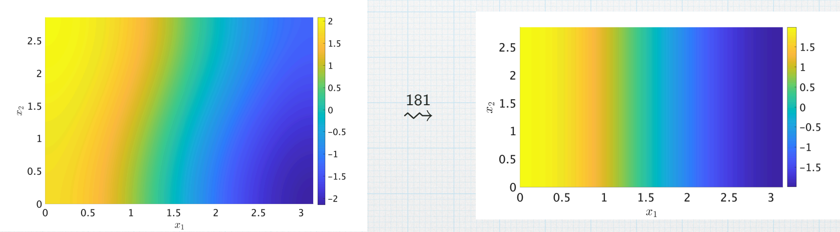
Thank you for kind attention!

Swift-Hohenberg equation (2D)

$$u_t = \sigma u - (1+\Delta)^2 u - u^3, \quad \sigma \in \mathbb{R}$$

with the Neumann b.c. on $[0,\pi] \times [0,\pi/1.1]$.

$\sigma = 3$, $\boldsymbol{N} = 13$, step size $h = 7.8125 \cdot 10^{-3}$, 181 steps:



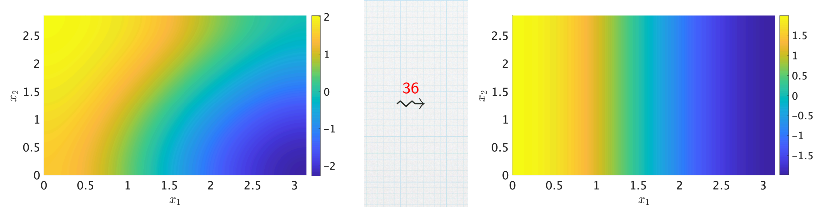
$$\sup_{t\in[0,1.4141]} \|u(t) - \bar{u}(t)\| \le 2.9081 \times 10^{-6}.$$

Swift-Hohenberg equation (2D)

$$u_t = \sigma u - (1+\Delta)^2 u - u^3, \quad \sigma \in \mathbb{R}$$

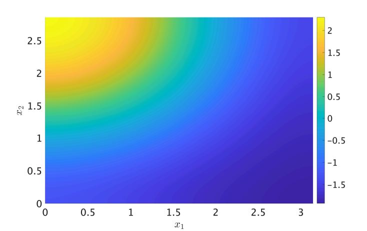
with the Neumann b.c. on $[0,\pi] \times [0,\pi/1.1]$.

$\sigma = 3$, $\boldsymbol{N} = 13$, step size $h = 6.25 \cdot 10^{-2}$, 36 steps:



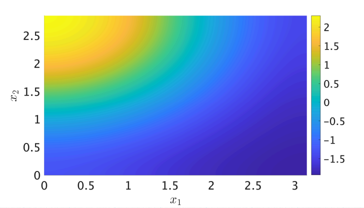
$$\sup_{t\in[0,2.25]} \|u(t) - \bar{u}(t)\| \leq 5.8744 \times 10^{-7}.$$

Initial profile.



$\overset{12}{\rightsquigarrow}$

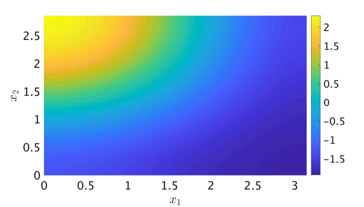
Profile of equilibrium.



$$\sup_{t\in[0,0.375]} \|u(t) - \bar{u}(t)\| \leq 1.2099 \times 10^{-6},$$

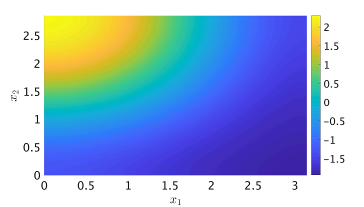
where the mesh size is $h = 3.125 \cdot 10^{-2}$.

Initial profile.



$\overset{21}{\rightsquigarrow}$

Profile of equilibrium.



$$\sup_{t\in[0,1.15]} \|u(t)-\bar{u}(t)\| \leq 3.5615 \times 10^{-7},$$

where the mesh size is $h = 6.25 \cdot 10^{-2}$.